

Positive solutions of nonlinear multi-point boundary value problems

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Received: 19 April 2017 / Accepted: 7 May 2018
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Abstract This paper deals with the existence of positive solutions of nonlinear differential equation

$$u''(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

subject to the boundary conditions

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} b_i u'(\xi_i),$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, and a_i, b_i satisfy $a_i, b_i \in [0, \infty)$, $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$. By using Schauder's fixed point theorem, we show that it has at least one positive solution if f is nonnegative and continuous. Positive solutions of the above boundary value problem satisfy the Harnack inequality

$$\inf_{0 \leq t \leq 1} u(t) \geq \gamma \|u\|_{\infty}.$$

Keywords Differential equation · Nonlinear boundary value problems · Positive solutions · Fixed point theorem

Mathematics Subject Classification 34B15 · 34B18

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1 Introduction

In this paper, we investigate the existence of positive solutions to boundary value problem (BVP)

$$u''(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad (1.1)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad (1.2)$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a_i, b_i \in [0, \infty)$, with $0 < \sum_{i=1}^{m-2} a_i < 1$, and $0 < \sum_{i=1}^{m-2} b_i < 1$.

During the last thirty years, multi-point BVPs have been studied extensively, for details, see [3–34, 36, 37] and the references therein. They describe many phenomena in the applied mathematical sciences, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases, and in concentration in chemical or biological problems, where only positive solutions are meaningful.

Dogan [5] investigated the existence of positive solutions to BVP

$$u'' + \lambda f(t, u) = 0, \quad 0 < t < 1,$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i),$$

where $\xi_i \in (0, 1)$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\alpha_i, \beta_i \in [0, \infty)$, λ is positive parameter. By using Krasnosel'skii's fixed point theorem, we proved sufficient conditions for the existence of at least one positive solution to the above BVP.

Feng and Ge [7] considered the existence of positive solutions for a semipositone second-order multi-point BVP

$$u'' + \lambda f(t, u) = 0, \quad 0 < t < 1,$$

$$u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i).$$

By using Krasnosel'skii's fixed point theorem, they found sufficient conditions that guarantee the existence of at least one positive solution.

Gupta [11] studied the existence of solutions for the generalized multipoint BVP

$$x'' = f(t, x(t), x'(t)) + r(t), \quad 0 < t < 1,$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(1) = \sum_{i=1}^{n-2} \beta_i x'(\eta_i),$$

where $0 < \xi_1 < \dots < \xi_{m-2} < 1$, $0 < \eta_1 < \dots < \eta_{n-2} < 1$, $\alpha_i, \beta_i \in \mathbb{R}$ and $(1 - \sum_{i=1}^m \alpha_i)(1 - \sum_{i=1}^n \beta_i) \neq 0$. He established some existence results for the above BVP.

Zhou and Cao [37] studied the existence of positive solutions for the BVP

$$\begin{aligned}
 u'' + q(t)f(t, u, u') &= 0, & 0 < t < 1, \\
 u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), & & u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),
 \end{aligned}$$

by using a fixed point theorem which is a generalization of the fixed point theorem of Leggett-Williams.

Motivated on these studies, the aim of this paper is to show the existence of positive solutions to the BVP (1.1) and (1.2). Our main results will depend on an application of a fixed point theorem in cones. To the best of our knowledge, there are still no results for the existence of positive solutions to BVP (1.1) and (1.2) by using the fixed point theorem.

We will assume that the following conditions are satisfied throughout this paper:

- (H1) $a_i, b_i \in [0, \infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$;
- (H2) $a : [0, 1] \rightarrow [0, \infty)$ is continuous and there exists $t_0 \in [0, 1]$ such that $a(t_0) > 0$;
- (H3) $f : [0, \infty) \rightarrow [0, \infty)$ is continuous.

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

2 Some Lemmas

Lemma 2.1 *Let $\left(1 - \sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{i=1}^{m-2} b_i\right) \neq 0$, $h \in C[0, 1]$. Then the following BVP*

$$u''(t) + h(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \tag{2.2}$$

has a unique solution

$$u(t) = - \int_0^t (t-s)h(s)ds + C_1 t + C_2, \tag{2.3}$$

where

$$C_1 = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds \right),$$

$$C_2 = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) h(s) ds \right. \\ \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds \right) \right].$$

Proof From (2.1) we have

$$u''(t) = -h(t).$$

Now, we prove that u satisfies the boundary conditions in (2.2). On the one hand, we get

$$u(0) = C_2 = \frac{\left(\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} sh(s) ds \right) \left(1 - \sum_{i=1}^{m-2} b_i \right) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_i h(s) ds}{1 - \sum_{i=1}^{m-2} a_i},$$

and

$$u(\xi_i) = \frac{\int_0^{\xi_i} sh(s) ds \left(1 - \sum_{i=1}^{m-2} b_i \right) + \int_{\xi_i}^1 \xi_i h(s) ds}{1 - \sum_{i=1}^{m-2} a_i}$$

so that

$$\sum_{i=1}^{m-2} a_i u(\xi_i) = \frac{\left(\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} sh(s) ds \right) \left(1 - \sum_{i=1}^{m-2} b_i \right) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_i h(s) ds}{1 - \sum_{i=1}^{m-2} a_i};$$

thus,

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i).$$

On the other hand, we can find

$$u'(1) = \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds}{1 - \sum_{i=1}^{m-2} b_i},$$

and

$$u'(\xi_i) = \frac{\int_{\xi_i}^1 h(s)ds}{1 - \sum_{i=1}^{m-2} b_i},$$

so that

$$\sum_{i=1}^{m-2} b_i u'(\xi_i) = \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s)ds}{1 - \sum_{i=1}^{m-2} b_i} = u'(1).$$

Therefore,

$$u'(1) = \sum_{i=1}^{m-2} b_i u'(\xi_i),$$

so that u given in (2.3) is a solution of BVP (2.1) and (2.2).

It is easy to see that BVP $u'' = 0$, $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, $u'(1) = \sum_{i=1}^{m-2} b_i u'(\xi_i)$ has only the trivial solution. Thus, u in (2.3) is the unique solution of BVP (2.1) and (2.2). This completes the proof of the lemma. \square

Lemma 2.2 Assume (H1) holds for $h \in C[0, 1]$ and $h(t) \geq 0$. Then, the unique solution u of BVP (2.1) and (2.2) satisfies $u(t) \geq 0$, for $t \in [0, 1]$.

Proof According to Lemma 2.1, we get

$$\begin{aligned} u(0) = C_2 &= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right) \right] \\ &= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^1 h(s)ds + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s)ds \right) \right] \\ &= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\left(1 - \sum_{i=1}^{m-2} b_i\right) \int_0^1 h(s)ds + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s)ds \right) \right] \\ &\geq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)h(s)ds \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 h(s) ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \Big] \\
& = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^1 s h(s) ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \right] \geq 0
\end{aligned}$$

and

$$\begin{aligned}
u(1) & = - \int_0^1 (1-s) h(s) ds + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds \right) \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) h(s) ds \right. \\
& \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds \right) \right] \\
& \geq - \int_0^1 (1-s) h(s) ds + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds \right) \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s) h(s) ds \right. \\
& \left. + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 h(s) ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \right] \\
& = - \int_0^1 (1-s) h(s) ds + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s) ds \right. \\
& \left. - \sum_{i=1}^{m-2} b_i \int_0^1 h(s) ds + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \right) \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^1 s h(s) ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \right] \\
& = \int_0^1 s h(s) ds + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds}{1 - \sum_{i=1}^{m-2} b_i} \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^1 s h(s) ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds \right] \geq 0.
\end{aligned}$$

If $t \in (0, 1)$, we have

$$\begin{aligned}
 u(t) &= -\int_0^t (1-s)h(s)ds + \frac{t}{1-\sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right) \\
 &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)ds \right. \\
 &+ \left. \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1-\sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right) \right] \\
 &\geq -\int_0^1 (1-s)h(s)ds + \frac{t}{1-\sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right) \\
 &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[-\sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)h(s)ds \right. \\
 &+ \left. \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1-\sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right) \right] \\
 &= \int_0^1 sh(s)ds + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s)ds}{1-\sum_{i=1}^{m-2} b_i} \\
 &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^1 sh(s)ds + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s)ds \right] \geq 0.
 \end{aligned}$$

So, $u(t) \geq 0, t \in [0, 1]$. □

Lemma 2.3 Assume (H1) holds, for $h \in [0, 1]$ and $h(t) \geq 0$. Then, the unique solution u of BVP (2.1) and (2.2) satisfies

$$\inf_{t \in [0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1-\sum_{i=1}^{m-2} a_i (1-\xi_i)}, \quad \|u\| = \max_{t \in [0,1]} |u(t)|.$$

Proof Taking the derivate of (2.3) with respect to t , we obtain

$$\begin{aligned}
 u'(t) &= -\int_0^t h(s)ds + C_1 \\
 &\geq -\int_0^1 h(s)ds + \frac{1}{1-\sum_{i=1}^{m-2} b_i} \left(\int_0^1 h(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{i=1}^{m-2} b_i \int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s) ds}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_0^1 h(s) ds - \sum_{i=1}^{m-2} b_i \left(\int_0^1 h(s) ds + \int_1^{\xi_i} h(s) ds \right)}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 h(s) ds}{1 - \sum_{i=1}^{m-2} b_i} \geq 0.
\end{aligned}$$

This implies that

$$\min_{t \in [0,1]} u(t) = u(0), \quad \|u\| = u(1).$$

It is easy to see that $u'(t_2) \leq u'(t_1)$ for any $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$. Hence, u' is a decreasing function on $[0, 1]$. This means that graph of u is concave down on $(0, 1)$. For each $i \in \{1, 2, \dots, m-2\}$, we have

$$\frac{u(1) - u(0)}{1 - 0} \geq \frac{u(1) - u(\xi_i)}{1 - \xi_i},$$

that is,

$$u(\xi_i) - \xi_i u(1) \geq (1 - \xi_i)u(0),$$

so that

$$\sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i \xi_i u(1) \geq \sum_{i=1}^{m-2} a_i (1 - \xi_i) u(0).$$

With the boundary condition $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we find

$$u(0) \geq \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i (1 - \xi_i)} u(1).$$

This completes the proof. □

3 Main results

The BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation

$$\begin{aligned}
 Fu(t) = & - \int_0^t (t-s)a(s)f(u(s))ds \\
 & + \frac{t}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
 & + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u(s))ds \right. \\
 & \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \right].
 \end{aligned}$$

Denote

$$K = \left\{ u : u \in C[0, 1], u \geq 0, \min_{0 \leq t \leq 1} u(t) \geq \gamma \|u\| \right\}.$$

It is obvious that K is a convex subset of $C[0, 1]$.

Lemma 3.1 $F : K \rightarrow K$ is completely continuous.

Proof Firstly, we show that $F(K) \subset K$.

For any $u \in K$, we have

$$\begin{aligned}
 (Fu)''(t) + a(t)f(u(t)) &= 0, & 0 < t < 1, \\
 (Fu)(0) = \sum_{i=1}^{m-2} a_i Fu(\xi_i), & & (Fu)'(1) = \sum_{i=1}^{m-2} b_i (Fu)'(\xi_i).
 \end{aligned}$$

And,

$$\begin{aligned}
 (Fu)''(t) &= -a(t)f(u(t)), \\
 (Fu)'(t) &= - \int_0^t a(s)f(u(s))ds + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds \right. \\
 & \quad \left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
 &\geq - \int_0^1 a(s)f(u(s))ds + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds \right. \\
 & \quad \left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
 &= \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(- \int_0^1 a(s)f(u(s))ds + \sum_{i=1}^{m-2} b_i \int_0^1 a(s)f(u(s))ds \right)
 \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \Big) \\
& = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\sum_{i=1}^{m-2} b_i \int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
& = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)f(u(s))ds \right. \\
& \quad \left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
& = \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)f(u(s))ds}{1 - \sum_{i=1}^{m-2} b_i}, \quad 0 \leq t \leq 1.
\end{aligned}$$

Since $(Fu)''(t) \leq 0$, Fu is concave. And since $(Fu)'(t) \geq 0$ on $[0, 1]$, we can obtain that $(Fu)(t)$ is nondecreasing on $[0, 1]$. On the other hand, it is similar to the proof of Lemma 2.2. Thus, $Fu \in K$. Moreover, we know that Fu satisfies (2.2). Hence, Lemma 2.3 implies

$$\inf_{t \in [0,1]} (Fu)(t) \geq \gamma \|Fu\|, \quad \text{for } u \in K,$$

i.e. $Fu \in K$. Therefore, we can find that $F(K) \subset K$.

Secondly, we show that F maps bounded set into itself. Suppose that $c > 0$ is a constant and $u \in \overline{K}_c = \{u \in K : \|u\| \leq c\}$. Note that the continuity of f guarantees that there is a $L > 0$ such that $f(u(t)) \leq L$ for $t \in [0, 1]$. Therefore,

$$\begin{aligned}
\|Fu\| & = \max_{t \in [0,1]} (Fu)(t) \\
& = - \int_0^1 (1-s)a(s)f(u(s))ds \\
& \quad + \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \\
& \quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u(s))ds \right. \\
& \quad \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \right] \\
& = \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[- \int_0^1 (1-s)a(s)f(u(s))ds + \sum_{i=1}^{m-2} b_i \int_0^1 (1-s)a(s)f(u(s))ds \right. \\
& \quad \left. + \int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right]
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\left(1 - \sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) a(s) f(u(s)) ds \right. \\
 & + \sum_{i=1}^{m-2} b_i \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) a(s) f(u(s)) ds + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 a(s) f(u(s)) ds \\
 & \left. - \sum_{i=1}^{m-2} a_i \xi_i \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s) f(u(s)) ds \right] \\
 = & \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[- \int_0^1 a(s) f(u(s)) ds + \int_0^1 s a(s) f(u(s)) ds \right. \\
 & + \sum_{i=1}^{m-2} b_i \int_0^1 a(s) f(u(s)) ds - \sum_{i=1}^{m-2} b_i \int_0^1 s a(s) f(u(s)) ds \\
 & \left. + \int_0^1 a(s) f(u(s)) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s) f(u(s)) ds \right] \\
 & + \frac{1}{\left(1 - \sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \xi_i a(s) f(u(s)) ds \right. \\
 & + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} s a(s) f(u(s)) ds + \sum_{i=1}^{m-2} b_i \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \xi_i a(s) f(u(s)) ds \\
 & - \sum_{i=1}^{m-2} b_i \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} s a(s) f(u(s)) ds + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 a(s) f(u(s)) ds \\
 & \left. - \sum_{i=1}^{m-2} a_i \xi_i \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s) f(u(s)) ds \right] \\
 = & \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\int_0^1 s a(s) f(u(s)) ds - \sum_{i=1}^{m-2} b_i \int_0^1 s a(s) f(u(s)) ds \right. \\
 & \left. + \sum_{i=1}^{m-2} b_i \int_0^1 a(s) f(u(s)) ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s) f(u(s)) ds \right] \\
 & + \frac{1}{\left(1 - \sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \xi_i a(s) f(u(s)) ds \right. \\
 & + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} s a(s) f(u(s)) ds - \sum_{i=1}^{m-2} b_i \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} s a(s) f(u(s)) ds \\
 & \left. + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 a(s) f(u(s)) ds \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \left[\left(1 - \sum_{i=1}^{m-2} b_i\right) \int_0^1 sa(s)f(u(s))ds \right. \\
&\quad \left. + \sum_{i=1}^{m-2} b_i \left(\int_0^{\xi_i} a(s)f(u(s))ds + \int_{\xi_i}^1 a(s)f(u(s))ds \right) - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right] \\
&\quad + \frac{1}{\left(1 - \sum_{i=1}^{m-2} a_i\right)\left(1 - \sum_{i=1}^{m-2} b_i\right)} \left[\left(1 - \sum_{i=1}^{m-2} b_i\right) \right. \\
&\quad \times \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} sa(s)f(u(s))ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \xi_i a(s)f(u(s))ds \\
&\quad \left. + \sum_{i=1}^{m-2} a_i \left(\int_0^{\xi_i} \xi_i a(s)f(u(s))ds + \int_{\xi_i}^1 \xi_i a(s)f(u(s))ds \right) \right] \\
&= \int_0^1 sa(s)f(u(s))ds + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)f(u(s))ds}{1 - \sum_{i=1}^{m-2} b_i} \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} sa(s)f(u(s))ds + \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_i a(s)f(u(s))ds}{1 - \sum_{i=1}^{m-2} b_i} \right] \\
&\leq L \left(\int_0^1 sa(s)ds + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)ds}{1 - \sum_{i=1}^{m-2} b_i} \right. \\
&\quad \left. + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} sa(s)ds + \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \xi_i a(s)ds}{1 - \sum_{i=1}^{m-2} b_i} \right] \right).
\end{aligned}$$

That is, $F\overline{K}_c$ is uniformly bounded.

Thirdly, for $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned}
&\left| (Fu)(t_1) - (Fu)(t_2) \right| \\
&= \left| - \int_0^{t_1} (t_1 - s)a(s)f(u(s))ds \right. \\
&\quad \left. + \frac{t_1}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \right. \\
&\quad \left. + \int_0^{t_2} (t_2 - s)a(s)f(u(s))ds \right. \\
&\quad \left. - \frac{t_2}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u(s))ds \right) \right| \\
&\leq L|t_1 - t_2| \left(\int_0^1 a(s)ds + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)ds}{1 - \sum_{i=1}^{m-2} b_i} \right) \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2.
\end{aligned}$$

Therefore, by applying the Arzela–Ascoli theorem [1], we can find that $F\overline{K_c}$ is relatively compact.

Finally, we claim that $F : \overline{K_c} \rightarrow K$ is continuous. Assume that $\{u_n\}_{n=1}^\infty \subset \overline{K_c}$ which converges to $u_0(t)$ uniformly on $[0, 1]$. Hence, $\{(Fu_n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on $[0, 1]$. The Arzela–Ascoli theorem [1] implies that there exists a uniformly convergent subsequence in $\{(Fu_n)(t)\}_{n=1}^\infty$. Let $\{(Fu_{n(m)})(t)\}_{m=1}^\infty$ be a subsequence which converges to $v(t)$ uniformly on $[0, 1]$. Observe that

$$\begin{aligned} (Fu_n)(t) &= - \int_0^t (t-s)a(s)f(u_n(s))ds \\ &+ \frac{t}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u_n(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u_n(s))ds \right) \\ &+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u_n(s))ds \right. \\ &\left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u_n(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u_n(s))ds \right) \right]. \end{aligned}$$

Inserting $u_{n(m)}$ into the above and then letting $m \rightarrow \infty$, we find

$$\begin{aligned} v(t) &= - \int_0^t (t-s)a(s)f(u_0(s))ds \\ &+ \frac{t}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u_0(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u_0(s))ds \right) \\ &+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left[- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(u_0(s))ds \right. \\ &\left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i} \left(\int_0^1 a(s)f(u_0(s))ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(u_0(s))ds \right) \right], \end{aligned}$$

where we have used Lebesgue’s dominated convergence theorem [2]. From the definition of F , we know that $v(t) = Fu_0(t)$ on $[0, 1]$. This shows that each subsequence of $\{Fu_n(t)\}_{n=1}^\infty$ uniformly converges to $(Fu_0)(t)$. Therefore, the sequence $\{(Fu_n)(t)\}_{n=1}^\infty$ uniformly converges to $(Fu_0)(t)$. This means that F is continuous at $u_0 \in \overline{K_c}$. So, F is continuous on $\overline{K_c}$ since u_0 is arbitrary. Thus, F is completely continuous. The proof is complete. \square

We can conclude by the following Schauder’s fixed point theorem.

Theorem 3.1 (Schauder’s Fixed Point Theorem) *Let K be a closed convex subset of the Banach space E . Suppose $F : K \rightarrow K$ and F is compact. (i.e., bounded sets in K are mapped into relatively compact sets). Then, F has a fixed point in K .*

Theorem 3.2 Suppose that conditions (H1), (H2) and (H3) are satisfied. Then the BVP (1.1) and (1.2) has at least one positive solution in K .

Proof It is a consequence of Lemma 3.1 and Theorem 3.1. \square

4 Uniqueness of positive solutions

We make the following assumptions on the nonlinearity f :

(S1) $f \in C([0, +\infty), [0, +\infty))$ and there is $p > 0$ such that $f(x) > 0$, $x \in (p, +\infty)$;

(S2) $f \in C^1([0, +\infty), [0, +\infty))$ and $f'(x) \leq 0$, $\forall x \geq 0$.

As a consequence of the Harnack inequality one gets.

Theorem 4.1 If (S1) holds, then the BVP (1.1) and (1.2) has at least one positive solution $u \in C^2([0, 1], [0, +\infty))$ with $u(t) > 0$, $t \in (0, 1)$.

The following result is a consequence of the strong maximum principle.

Lemma 4.1 ([35]) Suppose $g \in E$ and $g(t) \leq 0$, $\forall t \in [0, 1]$. If $w \in C^2[0, 1]$ solves

$$\begin{aligned} w'' + g(t)w &= 0, \\ w(0) &= \sum_{i=1}^{m-2} a_i w(\xi_i), \\ w'(1) &= \sum_{i=1}^{m-2} b_i w'(\xi_i), \end{aligned} \tag{4.1}$$

then $w(t) \equiv 0$.

Theorem 4.2 If (S1), (S2) hold, then the BVP (1.1) and (1.2) has one unique positive solution $u \in C^2([0, 1], [0, +\infty))$ with $u(t) > 0$, $t \in (0, 1)$.

Proof Theorem 4.1 asserts that the BVP (1.1) and (1.2) has at least one positive solution $u \in C^2([0, 1], [0, +\infty))$ with $u(t) > 0$, $t \in (0, 1)$. Suppose $u \in C^2([0, 1], [0, +\infty))$ and $v \in C^2([0, 1], [0, +\infty))$ are two positive solutions of BVP (1.1) and (1.2). Let

$$\begin{aligned} w(t) &:= u(t) - v(t), \\ g(t) &:= \begin{cases} \frac{f(u(t)) - f(v(t))}{w(t)}, & w(t) \neq 0, \\ f'(u(t)), & w(t) = 0. \end{cases} \end{aligned}$$

Then $w \in C^2([0, 1], [0, +\infty))$ and $g \in E$. By (S2), $g(t) \leq 0$, $\forall t \in [0, 1]$. Obviously, w and g satisfy (4.1). Lemma 4.1 implies that $w(t) \equiv 0$ and thus $u(t) \equiv v(t)$. This completes the proof. \square

Acknowledgements The author would like to thank the anonymous referees and editor for their helpful comments and suggestions.

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