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ORIGINAL PAPER

Rings with modules having a restricted injectivity domain

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Abstract
We introduce modules whose injectivity domains are contained in the class of modules with zero radical and call them working-class. This notion gives a generalization of poor modules that have minimal injectivity domain. Semisimple working-class modules always exist for arbitrary rings whereas their predecessors do not. We investigate the rings over which every module is either injective or working-class. Right weakly V-rings are examples of these rings. Moreover, we study the existence of working-class simple modules and show that if there is a projective working-class simple right module, then the ring is a right GV-ring.

Keywords Injective module · Poor module · Working-class module · Good ring · WV-ring

Mathematics Subject Classification 16D50 · 16D70

1 Introduction

Throughout this study, all rings are associative with an identity element and all modules are right and unital. Let R be a ring and M be a right R-module. We denote the socle and the radical of M by Soc M and Rad M, respectively. $J(R)$ stands for the Jacobson

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radical of R. For terminology and notations used in this paper, we refer the reader to [6,8,11,15].

Poor modules are introduced in [1] as the ones having their injectivity domain minimal possible, namely the class of all semisimple modules, where injectivity domain of a module M is the class $\mathfrak{In}^{-1}(M) = \{N \in \mathscr{M}od\text{-}R : M \text{ is } N\text{-injective}\}$. This definition gives a natural opposite to injectivity of modules, since only injective modules have the class of all modules as their injectivity domain. Recently, many studies have been conducted concerning poor modules along with their generalizations and restrictions (see [2–4,7,9]).

In [7], modules whose injectivity domain is contained in the class of all pure-split modules are introduced as *impecunious* modules, where a module M is called pure-split if all its pure submodules are direct summands of M. Starting point of defining such modules was the fact that semisimple modules are pure-split and therefore a generalization of poor modules is obtained.

Another property that not only semisimple modules admit, in general, is having zero radical. With this idea in mind, we expand the restriction in definition of poor modules and consider modules with their injectivity domain contained in the class of all modules with zero radical. We call a module M *working-class* if $\mathfrak{In}^{-1}(M) \subseteq \{N \in \mathscr{M}od\text{-}R : \operatorname{Rad} N = 0\}$. This definition gives a natural generalization by the property of semisimple modules we mentioned. Indeed, this generalization is proper (see Example 2).

In Sect. 2, we consider the rings over which every right module is working-class and show that such rings are exactly right V-rings, that is rings over which all simple modules are injective (see Theorem 1). Along with some properties of working-class modules, we show that every ring has a semisimple working-class module. Modules for which every cyclic submodule is semisimple is clearly itself semisimple and this fact is used frequently in papers concerning poor modules. This argument is not true when the term semisimplicity is replaced by having zero radical. A ring R is called *right good* if $f(\operatorname{Rad} M) = \operatorname{Rad} f(M)$ for every homomorphism $f\colon M \longrightarrow N$ of right R-modules M and N. Over such rings, we have that a module has zero radical if and only if so do all its cyclic submodules (see Lemma 4). We also show that over a commutative hereditary Noetherian ring, a module is working-class if and only if it is poor so that being working-class and poor are equivalent conditions for an abelian group.

In Sect. 3, we introduce the rings over which every module is either working-class or injective and call such condition property P. Clearly, if the ring has no middle class, that is if every module is either poor or injective, then it has property P. By Theorem 1, the converse of this statement does not hold, in general. However, these two conditions are equivalent for a semilocal ring. As generalizations of right V-rings, a ring R is called *a right weakly-V-ring* (WV-ring for short) if every simple right R-module is R/I-injective for any right ideal I such that R/I is proper and it is called *a right GV-ring* if every simple right R-module is either injective or projective. We follow footsteps of results from [4,9] in investigating rings with property P and their relation to rings with no middle class. In particular, we show that a right WV-ring has property P and if, moreover, it is right semiartinian, then it is either a right V-ring or a right Artinian ring (see Theorem 6 and Corollary 7). We also prove that for a right nonsingular, right Noetherian and right GV-ring R, every simple right R-module is either working-class or injective if and only if either R is a right V-ring or there is a ring decomposition $R = S \oplus T$, where S is a semisimple Artinian ring and right socle of T is nonzero working-class homogeneous (see Theorem 8). We end this section by showing that a commutative Noetherian ring with property P is Artinian so that such rings also admit the property we mentioned for semilocal rings (see Theorem 9).

In the last section, we focus our attention on simple working-class modules. We introduce rings over which every simple right module is working-class and call such rings right SW. We show that if a ring R has a projective working-class simple module, then R_R is working-class and R is a right GV-ring (see Proposition 8). In Theorem 11, a right SW-ring is shown to either be a right V-ring or have all its simple right modules singular. As an application to one of the extensively studied rings in the literature, being right δ-perfect and right perfect are equivalent conditions for a right SW-ring which is not a right V-ring (see Corollary 10). The last result states that a commutative Noetherian ring is an SW-ring if and only if it is either local or semisimple Artinian (see Theorem 12).

2 Working-class modules

It is well known that $\operatorname{Rad} N = 0$ for a semisimple module N. Motivated by this fact, we generalize poor modules to working-class modules using injectivity domains and investigate some properties of these modules. In particular, we prove that every abelian working-class group is poor.

Definition 1 Let R be a ring and M be an R-module. M is called *working-class* if the injectivity domain $\mathfrak{In}^{-1}(M)$ of M is contained in the class of all modules with zero radical, that is, $\mathfrak{In}^{-1}(M) \subseteq \{ N \in \mathscr{M}od\text{-}R \,:\, \operatorname{Rad} N = 0 \}$.

Recall from [1, Remark 2.3] that a ring R is semisimple Artinian if and only if every right R-module is poor. As a proper generalization of semisimple Artinian rings, a ring R is called *a right V-ring* if every simple right R-module is injective ([11, 6.1]). It is possible to give a characterization of right V-rings via working-class modules.

Theorem 1 *The following statements are equivalent for a ring R.*

(1) 0 *is a working-class R-module.*
(2) *there exists an injective working-class R-module.*
(3) *R is a right V-ring.*
(4) *every R-module is working-class.*
(5) *every factor module of a working-class R-module is working-class.*
(6) *every direct summand of a working-class R-module is working-class.*

Proof (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1) are clear.

For (2) $\Rightarrow$ (3), let M be an injective working-class module. Then $\mathscr{M}od\text{-}R = \mathfrak{In}^{-1}(M) = \{ N \in \mathscr{M}od\text{-}R \,:\, \operatorname{Rad} N = 0 \}$. It follows from [11, Theorem 6.1] that R is a right V-ring.

For (3) $\Rightarrow$ (4), suppose that M is N-injective for an R-module N. Since R is a right V-ring, applying [11, Theorem 6.1] once again, we obtain that $\operatorname{Rad} N = 0$. Hence M is working-class. □

In [9], the authors investigated the existence of semisimple poor modules and characterized such rings (see [9, Theorem 1]). In the case of working-class modules the situation is much simpler as they always do exist. Before stating the result let us recall some information on small submodules. A proper submodule N of a module M is called *small* in M and denoted by $N \ll M$, if $N + L \neq M$ for every proper submodule L of M. It is well known that $\operatorname{Rad} M$ is the sum of all small submodules of M and if $\alpha \colon M \longrightarrow K$ is a homomorphism of right R-modules M and K, then $\alpha(N) \ll K$ for every small submodule N of M.

Theorem 2 *Every ring has a semisimple working-class module.*

Proof Let R be a ring and $\{S_\gamma : \gamma \in \Gamma\}$ be a complete set of representatives of isomorphism classes of non-injective simple right R-modules. Then, $S = \bigoplus_{\gamma \in \Gamma} S_\gamma$ is working-class. To see this, suppose that S is N-injective for some right R-module N with $\operatorname{Rad} N \neq 0$. For $0 \neq n \in \operatorname{Rad} N$, nR has a maximal submodule M and since $nR \ll N$, nR/M is a non-injective simple right R-module. Therefore, $nR/M \cong S_{\gamma_0}$ for some $\gamma_0 \in \Gamma$ so that there is a nonzero homomorphism $f \colon nR \longrightarrow S$. Since S is N-injective, there is a homomorphism $g \colon N \longrightarrow S$ with $g|_{nR} = f$. Now, we have $f(nR) = g(nR) \subseteq g(\operatorname{Rad} N) \subseteq \operatorname{Rad} g(N) \subseteq \operatorname{Rad} S = 0$ which is a contradiction. Hence S is working-class. □

It turns out that the proof for Theorem 2 can be used to find more working-class modules for any ring. Proofs for the following facts adhere to similar ideas used in proving Theorem 2 and therefore are omitted.

Example 1 Let R be any ring, $M = \bigoplus_{A \in \Gamma} A$ and $N = \prod_{A \in \Gamma} A$, where Γ is a complete set of representatives of isomorphism classes of either one of the following classes of modules:

(1) simple modules,
(2) non-injective cyclic modules with zero radical,
(3) cyclic modules with zero radical.

Then M and N are working-class.

Working-class modules admit some basic properties as their predecessors do.

Lemma 1 *For any index set I, $\prod_{i \in I} M_i$ is working-class if and only if M_i is working-class for $i \in I$.*

Proof It follows from [15, 16.1 (1)]. □

Corollary 1 *Let R be a ring. Then, R_R is working-class if and only if every direct product of copies of R_R is working-class.*

Lemma 2 *Let M be a module. If a direct summand of M is working-class, then so is M.*

Proof Suppose that M is N-injective for a module N. Let $M = A \oplus B$, where A is working-class. Therefore A is N-injective. By the assumption, we get $\operatorname{Rad} N = 0$. So M is working-class. □

Note that Lemma 2 together with Theorem 2 can also be used in proving the modules given in Example 1 are working-class.

Corollary 2 *Let R be a ring. R_R is working-class if and only if every free R-module is working-class.*

The converse of Lemma 2 need not be true in general (see [1, Example 3.6] and [2, Corollary 3.2]), but it holds in a special case as it does in [7, Lemma 3.2] for impecunious modules.

Lemma 3 *Let M be a module and $M = N \oplus E$ for some submodules N and E with E injective. If M is working-class, then so is N.*

In [3], a module M is called *pauper* if M is poor and has no proper poor direct summand. The proof for the next result follows the one for [3, Proposition 2.1] and is given for the sake of completeness.

Proposition 1 *Let R be an arbitrary ring and M be a working-class R-module. If M has finite uniform dimension, then M has a working-class direct summand K such that K has no proper working-class direct summands.*

Proof If M has a working-class direct summand, then there are nonzero submodules M_1 and N_1 of M such that $M = M_1 \oplus N_1$, where M_1 is working-class. If M_1 is working-class with no proper direct summands, then we are done. Otherwise, there are nonzero submodules M_2 and N_2 of M_1 with M_2 working-class such that $M_1 = M_2 \oplus N_2$. Since M has finite uniform dimension and $M > M_1 > M_2$ is a chain of direct summands of M, by the same procedure we shall obtain a working-class submodule K of M such that K has no working-class proper direct summands after finitely many steps. □

In almost all the papers concerning poor modules, the authors make use of [15, 16.3] and the fact that a module N is semisimple if every cyclic submodule of N is semisimple. The second argument is not applicable in the case of working-class modules, since a module might have a nonzero radical even if all its cyclic submodules have zero radical. Rational numbers as an abelian group is an example of such modules. Nevertheless, there is a class of rings that give us the opportunity to follow a similar path.

Following [15, 23.7], a ring R is called *a right good ring* if every right R-module M is good, that is, $f(\operatorname{Rad} M) = \operatorname{Rad} f(M)$ for every homomorphism $f : M \longrightarrow N$ of right R-modules M and N. Right V-rings, commutative semiregular rings and commutative max rings are examples of right good rings. In particular, a commutative max ring with nonzero radical is a good ring which is not a V-ring.

Lemma 4 *Let R be a right good ring and M be an R-module. Then,* $\operatorname{Rad} M = 0$ *if and only if* $\operatorname{Rad} mR = 0$ *for every element $m \in M$.*

Proof We can write $M = \sum_{m\in M} mR$. For $A = \bigoplus_{m\in M} mR$, there is an epimorphism $f : A \longrightarrow M$. Since R is a right good ring, it follows from [15, 21.6 (5)] that $\operatorname{Rad} M = \operatorname{Rad} f(A) = f(\operatorname{Rad} A) = f(\operatorname{Rad}(\bigoplus_{m\in M} mR)) = f(\bigoplus_{m\in M} \operatorname{Rad} mR) = f(0) = 0$. □

The following result is a consequence of Lemma 4 and plays a key role in our work. Note that we shall freely use this fact without reference in this paper for right good rings.

Corollary 3 *Let R be a right good ring and M be an R-module. Then, M is working-class if and only if* $\operatorname{Rad} xR = 0$ *for every cyclic R-module $xR \in \mathfrak{In}^{-1}(M)$.*

Proposition 2 *Let R be a right good ring and $\{C_i\}_{i\in I}$ be a complete of representatives of isomorphism classes of cyclic right R-modules with nonzero radical. Then, $M_1 = \prod_{i\in I} \operatorname{Rad} C_i$ and $M_2 = \bigoplus_{i\in I} \operatorname{Rad} C_i$ are working-class. If R is a right semiartinian ring, then $M_3 = \prod_{i\in I} \operatorname{Soc} C_i$ and $M_4 = \bigoplus_{i\in I} \operatorname{Soc} C_i$ are working-class.*

Proof Let M_1 (or M_2) be N-injective for an R-module N. Assume to the contrary that $\operatorname{Rad} N \neq 0$. By Lemma 4, there is a nonzero element $n \in N$ such that $\operatorname{Rad} nR \neq 0$ and so we can write $nR \cong C_{i_0}$ for some $i_0 \in I$. Therefore $\operatorname{Rad} nR \cong \operatorname{Rad} C_{i_0}$. Since M_1 is N-injective, it follows from [15, 16.1 (1) and 16.3] that $\operatorname{Rad} nR$ is nR-injective. This is a contradiction. Hence M_1 is working-class.

Let R be a right semiartinian ring. Then every right R-module has an essential socle and so a cyclic right R-module xR with nonzero radical has a proper socle. Hence by adapting the previous argument we can prove similarly that M_3 and M_4 are working-class. □

In [3], the authors obtained a generalization for counterpart of Lemma 2 for poor modules over right Noetherian rings (see [3, Theorem 4.1]). The proof for the following result is almost the same as the one for [3, Theorem 4.1] and it is omitted to avoid repetition.

Theorem 3 *Let R be a right good and right Noetherian ring, M be a right R-module and L be a pure submodule of M. If L is working-class, then so is M.*

A ring R is said to be *local* if $R/J(R)$ is simple, and it is said to be *semilocal* if $R/J(R)$ is semisimple. Every semilocal ring is a right good ring. Now, we prove that over a semilocal ring every working-class module is poor. First we need the following well known fact that we include here for completeness.

Lemma 5 *Let R be a semilocal ring and M be a right R-module with zero radical. Then M is semisimple.*

Proof Let $U \leq M$. Since R is semilocal, there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq \operatorname{Rad} M$ by [6, 17.2 and 17.22 (1)]. Therefore, $U \cap V \subseteq \operatorname{Rad} M = 0$ and so $M = U \oplus V$. It means that M is semisimple. □

Corollary 4 *Let R be a semilocal ring and M be a right R-module. Then, M is working-class if and only if it is poor.*

Proof Let M be a working-class module and $xR \in \mathfrak{In}^{-1}(M)$. Then, M is xR-injective and so $\operatorname{Rad} xR = 0$ by the hypothesis. Since R is semilocal, it follows from Lemma 5 that xR is semisimple. Hence M is poor. □

Observe from Corollary 4 that, for $n > 1$, every working-class $\mathbb{Z}/n\mathbb{Z}$-module is poor. The following result is crucial in our work.

Theorem 4 *Let R be a right good ring such that $J(R)$ is simple and essential right ideal of R. Then, every non-injective right R-module is working-class.*

Proof Let M be a non-injective right R-module and M be nR-injective for a cyclic module nR. Since M is not injective, $\operatorname{ann}_r(n) \neq 0$ and so $J(R) \subseteq \operatorname{ann}_r(n)$ by the assumption. This implies that $\operatorname{Rad} nR = 0$. It follows from Corollary 3 that M is working-class. □

In [11, p. 57], a ring R is called *a right PCI-ring* if each proper cyclic right R-module is injective. Clearly, every right PCI-ring is a right V-ring. So, by Theorem 1, every right module over a right PCI-ring is working-class. Note that a commutative ring R is a V-ring if and only if it is von Neumann regular.

Example 2 (1) Consider the non-Noetherian commutative ring $S = \prod_{i=1}^{\infty} F_i$, where $F_i = F$ is any field. Let R be the subring of S consisting of all sequences $(r_n)_{n\in\mathbb{N}}$ such that there exist $r \in F, m \in \mathbb{N}$ with $r_n = r$ for all $n \geq m$. Then R is a von Neumann regular ring which is not semisimple Artinian. Therefore, $\operatorname{Soc} R_R$ is a maximal submodule of M and so $R/\operatorname{Soc} R_R$ is a simple R-module. Since R is a V-ring, $R/\operatorname{Soc} R_R$ is working-class by Theorem 1. On the other hand, $R/\operatorname{Soc} R_R$ is not poor.

(2) Let R be a right PCI-domain. It follows from [11, Theorem 5.3] that R is right Noetherian and right hereditary. Suppose that M is a semisimple right R-module as in [1, Proposition 3.4]. Then, M is a working-class module which is not poor.

By Example 2 (2), we see that over a right hereditary and right Noetherian domain a working-class module need not be poor. The following theorem shows that over a commutative hereditary Noetherian ring, every working-class module is poor.

Theorem 5 *Let R be a commutative hereditary Noetherian ring and M be an R-module. Then the following statements are equivalent.*

1. *M is poor.*
2. *M is working-class.*
3. *For every non-injective simple right R-module S, M has a direct summand isomorphic to S.*

Proof (1) $\Rightarrow$ (2) is clear and (3) $\Rightarrow$ (1) follows from [3, Theorem 4.8].

(2) $\Rightarrow$ (3) Assume that there is a non-injective simple R-module S such that M has no direct summand isomorphic to S. Since S is not injective, there is an R-module of length two such that $S \trianglelefteq N$ and $N/S \cong S$. Therefore S is a small submodule of N and so $\operatorname{Rad} N \neq 0$. By the first part of the proof for [3, Theorem 4.8], M is N-injective. This is a contradiction since M is working-class and $\operatorname{Rad} N \neq 0$. □

The characterization of poor abelian groups is given in [2, Theorem 3.1]. The same characterization holds for working-class abelian groups as Theorem 5 indicates.

Corollary 5 *An abelian group is working-class if and only if it is poor.*

3 Rings over which modules have restricted injectivity domains

In [9], rings with no right middle class are investigated and their complete characterization along with some results on rings with no right simple middle class is given in [4]. In this section, we investigate the rings over which every right module is either working-class or injective. Let us first shorten the notation with the following definition.

Definition 2 We say that a ring R has the property P if every right R-module is either working-class or injective.

Rings with no right middle class clearly possess the property P. A right V-ring which is not semisimple Artinian is an example of a ring with the property P that does not have no right middle class by Theorem 1.

The following result follows from Corollary 4.

Corollary 6 *Let R be a semilocal ring. Then, R has the property P if and only if it is a ring with no right middle class.*

Lemma 6 *Having property P is preserved under taking factor rings.*

Proof Let R be a ring with property P and I be an ideal of R. Let M be an R/I-module which is not working-class. Then M is N-injective for some R/I-module with $\operatorname{Rad} N \neq 0$. Then M_R is N_R-injective and $\operatorname{Rad} N_R \neq 0$. Since M_R is not working-class, it is injective. Hence M is injective as an R/I-module. ☐

Lemma 7 *Let R have a ring decomposition $R = S \oplus T$ with S semisimple Artinian. Then T has property P, then so does R.*

Proof Let M be right R-module which is not working-class. Then M is N-injective for some right R-module N with $\operatorname{Rad} N \neq 0$. We have that $M = MS \oplus MT$, $N = NS \oplus NT$ and MT is NT-injective as both right R-modules and right T-modules. Since $\operatorname{Rad} NT \neq 0$ and T has property P, MT is an injective right T-module and also an injective right R-module. Similarly, MS is an injective right R-module. Hence M is an injective right R-module. ☐

Remark 1 It is easy to see from the previous proof that if every simple right T-module is either working-class or injective, then simple right R-modules have the same property.

In [10], a ring R is *a right weakly-V-ring* (WV-ring for short) if every simple right R-module is R/I-injective for any right ideal I such that R/I is proper. Every right V-ring is a right WV-ring, but a right WV-ring need not be a right V-ring. For example, for any prime integer $p \in \mathbb{Z}$ the ring $\mathbb{Z}_{p^2}$ is a right WV-ring which is not a right V-ring.

Proposition 3 *Let R be a right WV-ring. Then it is a right good ring.*

Proof If R is a right V-ring, then it is clear. Assume that R is not a right V-ring. By [10, Corollary 4], $R/J(R)$ is a right V-ring. It follows from [15, pp. 192 and 23.7] that R is a right good ring. ☐

Theorem 6 *Let R be a right WV-ring. Then, R has the property P.*

Proof If R is a right V-ring, it follows from Theorem 1 that every module is working-class. Suppose that R is not a right V-ring. Then, $J(R)$ is simple and R_R is uniform by [11, Theorem 6.6 and Lemma 6.12]. It follows from Proposition 3 that R is a right good ring. Applying Theorem 4, we deduce that R has the property P. □

As another generalization of a right V-ring, a ring R is called *a right GV-ring* if every simple right R-module is either injective or projective. The proof for the following result is straightforward, we give it for further references.

Lemma 8 *A right GV-ring with zero right socle is a right V-ring.*

Now, we follow the footsteps of results from [4,9] in investigating the rings with property P. For this reason, some of the proofs for the following results are shorten.

Proposition 4 *Let R be a non-right GV-ring with property P. Then the following conditions hold.*

(1) *Every nonsingular right R-module is injective (hence semisimple).*
(2) *The second singular submodule splits in any right R-module.*

Proof To show (1), let R be a non-right GV-ring with property P and N be a nonsingular right R-module. By [13, Proposition 2.4], there is a singular right R-module M with $\mathrm{Rad}\, M \neq M \cap \mathrm{Rad}\, E(M)$ and so $\mathrm{Rad}\, E(M) \neq 0$. Since N is $E(M)$-injective and R has property P, N is injective.

(2) follows from the proof for [9, Lemma 2(ii)], since nonsingular right R-modules are also injective in this case. □

Proposition 5 *Let R be a ring with property P and essential and singular right socle. Then either R is indecomposable as a ring or $J(R) = 0$.*

Proof Assume that $R = R_1 \oplus R_2$ for nonzero ideals R_1 and R_2. Following the proof for [9, Lemma 3], either R_1 is semisimple or $\mathrm{Rad}\, R_2 = 0$ which implies that $\mathrm{Rad}\, R_2 = 0$. Similarly, we have that $\mathrm{Rad}\, R_1 = 0$. Hence $J(R) = 0$. □

Lemma 9 *Let R be a ring with property P. If there is a Noetherian right R-module with nonzero radical, then R is right Noetherian.*

Proof Let N be a Noetherian right R-module with nonzero radical and $\{E_i : i \in \Gamma\}$ be a family of injective right R-modules. Then, by [8, 2.5 p.10], $\bigoplus_{i\in\Gamma} E_i$ is N-injective. Since R has property P and $\mathrm{Rad}\, N \neq 0$, $\bigoplus_{i\in\Gamma} E_i$ is injective. Hence R is right Noetherian. □

Proposition 6 *A right semiartinian ring with property P and singular right socle is right Artinian.*

Proof Let R be a right semiartinian ring with property P and singular right socle. Since R is semiartinian, there is a simple right ideal S of R. S is not injective, otherwise it would be a direct summand of R which contradicts with the fact that S is singular. Since R is right semiartinian, $E(S)/S$ has a simple submodule. So there is a maximal submodule S' of $E(S)$ containing S. S' is Noetherian and $\mathrm{Rad}\, S' \neq 0$, since $S \ll S'$. Then R is right Noetherian by Lemma 9. Hence R is right Artinian. □

Theorem 7 *A right semiartinian ring R with property P is either a right V-ring or a right Artinian ring.*

Proof Assume that R is not a right V-ring. Then there is a simple right R-module S which is not injective. Then there is a Noetherian right R-module with nonzero radical as in the proof for Proposition 6 and R is right Noetherian by Lemma 9. Hence R is right Artinian. □

The next result is a direct consequence of Theorems 6 and 7.

Corollary 7 *A right semiartinian and a right WV-ring is either a right V-ring or a right Artinian ring.*

Theorem 8 *Let R be a right nonsingular, right Noetherian and right GV-ring. Then, every simple right R-module is either working-class or injective if and only if either R is a right V-ring or there is a ring decomposition $R = S \oplus T$, where S is a semisimple Artinian ring and right socle of T is nonzero working-class homogeneous.*

Proof Let R be a right nonsingular, right Noetherian ring and every simple right R-module be either working-class or injective.

For necessity, let S_1 and S_2 be two orthogonal semisimple right R-modules. We will show that at least one of S_1 and S_2 is injective. Suppose that S_1 is not injective. Then there is a non-injective simple direct summand of S_1 so that S_1 is working-class by assumption on R and Lemma 2. Following the proof for [4, Theorem 3.3] or [9, Lemma 8], we have that S_1 is $E(S_2)$-injective. Therefore, Rad $S_2 = 0$ which implies that every simple submodule of S_2 is injective as it is a direct summand of $E(S_2)$. Since R is right Noetherian, $E(S_2) = S_2$ and so S_2 is injective.

Since one of any two orthogonal semisimple submodules of Soc R_R is injective, we can write Soc $R_R = S \oplus H$ where S is injective and H is homogeneous and working-class. Since S is injective, $R = S \oplus T$ for some right ideal T of R. Following the proof for [4, Theorem 3.3], we obtain a ring decomposition $R = S \oplus T$ where S is semisimple Artinian and Soc $T_T \cong H$ is either zero or homogeneous and working-class.

Since T is also a GV-ring, Soc $T_T = 0$ implies by Lemma 8 that R is a right V-ring. If Soc $T_T \neq 0$, then we have the required decomposition.

For sufficiency, let $R = S \oplus T$, where S is a semisimple Artinian ring and right socle of T is nonzero working-class homogeneous. Soc T_T is not injective, otherwise R would be a right V-ring. Since R is a Noetherian GV-ring , Soc T_T is finitely generated homogeneous projective. A simple projective right T-module is isomorphic to the simple right ideal of T and so it is working-class. Remark 1 completes the proof. □

Note that necessity part of previous result remains true for rings with property P.

Corollary 8 *Let R be a right nonsingular, right Noetherian and right GV-ring. If R has property P, then either R is a right V-ring or it has a ring decomposition $R = S \oplus T$, where S is a semisimple Artinian ring and right socle of T is nonzero working-class homogeneous.*

In the following theorem, we completely determine the structure of commutative Noetherian rings with the property P.

Theorem 9 *Let R be a commutative Noetherian ring with the property P. Then, R is an Artinian ring. In this case, R has no middle class.*

Proof If R is a V-ring, then it is a von Neumann regular ring because R is commutative. Therefore, R is a semisimple Artinian ring by the assumption. Suppose that R is not a V-ring. By [14, Theorem 4.6], it suffices to show that every proper prime ideal of R is maximal. Let S be a proper prime ideal of R. Since R is not a V-ring, there exists a maximal ideal T of R such that R/T is not injective. It follows that $\operatorname{Rad} E(R/T) \neq 0$. If $S = T$, then the proof is clear. Let $S \neq T$. By [14, Proposition 4.21], we have that $\operatorname{Hom}(E(R/T), E(R/S)) = 0$ and so R/S is $E(R/T)$-injective. Thus R/S is injective since R has the property P and $\operatorname{Rad} E(R/T) \neq 0$. Then R/S is a field, since it is a self-injective integral domain. Hence S is maximal. □

4 Rings over which simple modules are working-class

On the existence of simple working-class modules for arbitrary rings we have an immediate answer. It follows from Theorem 5 that there are no simple working-class modules over a non-local Dedekind domain (for example the ring $\mathbb{Z}$ of integers). In this section, we consider the opposite case.

In [1], a ring R is called *a right simple-destitute* if every simple right R-module is poor. It is shown in [1, Theorem 5.2] that every local Artinian ring is a right simple-destitute ring. Adapting the concept of right simple-destitute rings, we say that a ring R is *a right SW-ring* if every simple right R-module is working-class.

By Theorem 1, every right V-ring is a right SW-ring, but a right SW-ring need not be a right V-ring. To see this, one can consider the ring $R = \mathbb{Z}/4\mathbb{Z}$. It follows from Example 2 (1) that the class of right SW-rings properly contains the class of right simple-destitute rings.

Theorem 10 *If a ring R has only one simple right module (up to isomorphism), then it is a right SW-ring. In particular, R is simple-destitute in case it is local.*

Proof First part of the theorem is a consequence of Example 1 (1) and the second assertion follows from Corollary 4. □

Now, we shall prove that every simple module over a right SW-ring which is not a right V-ring is singular. First we need the following lemma. Recall that two modules are called *orthogonal* if they have no nonzero isomorphic submodules.

Lemma 10 *Let R be a ring and M and N be orthogonal semisimple right R-modules with M projective and working-class. If N is finitely generated or R is a right Noetherian ring, then N is injective.*

Proof By the proof for [7, Theorem 4.3], M is $E(N)$-injective where $E(N)$ is the injective hull of N. Then $\operatorname{Rad} E(N) = 0$, since M is working-class. Let S be any simple submodule of $E(N)$. It follows that S is a direct summand of $E(N)$. So S is injective. It means that every simple submodule of N is injective. If N is finitely generated, then it is injective as a direct sum of finitely many injective modules.

Let R be a right Noetherian ring. Then, any direct sum of injective modules is injective and so the semisimple module N is injective as a direct sum of injective simple modules. ☐

Proposition 7 *Let R be a ring which is not a right V-ring and M be a projective working-class simple R-module. Then any simple R-module is either working-class or injective.*

Proof It is a consequence of Lemma 10. ☐

Proposition 8 *Suppose that a ring R has a projective working-class simple module. Then, R_R is working-class and R is a right GV-ring.*

Proof Let M be a projective working-class simple R-module. Then there exists a maximal right ideal T of R such that $M \cong R/T$. So the simple working-class module is isomorphic to a direct summand of R_R. It follows from Lemma 2 that R_R is working-class.

Let S be any simple R-module. If $S \cong M$, then S is projective. Suppose that $\mathrm{Hom}(S, M) = 0$. It means that S is orthogonal to M. By Lemma 10, we obtain that S is injective. Hence R is a right GV-ring. ☐

It is well known that any simple module is singular or projective. Using this fact, we can prove the following theorem.

Theorem 11 *Let R be a right SW-ring. Then, either R is a right V-ring or every simple R-module is singular.*

Proof If R is not a right V-ring, it follows from Theorem 1 that R has no injective simple working-class module. Suppose that there exists a projective simple R-module S. Let N be any simple R-module. Since R is a right SW-ring, N is working-class. If S is not isomorphic to N, then S and N are orthogonal and so N is injective by Lemma 10. This is a contradiction. Therefore, $S \cong N$. It means that every simple R-module is projective. Then R is semisimple Artinian, a contradiction. Hence every simple R-module is singular. ☐

Corollary 9 *Let R be a right SW-ring which is not a right V-ring. Then the following statements hold.*

(1) *Any simple module is neither injective nor projective.*
(2) *R is not a right GV-ring.*
(3) $\mathrm{Soc}\, R_R \subseteq J(R)$.
(4) $\mathrm{Soc}\, R_R \subseteq Z(R_R)$.
(5) *Every maximal right ideal of R is essential in R_R.*

Proof Since R is a right SW-ring, all simple modules are working-class.

(1) By Theorem 11, a simple module is not projective. If a simple module S is injective, it follows from Theorem 1 that R is a right V-ring. This contradicts with the assumption.

(2) It follows from (1).

(3) Let I be a minimal right ideal of R. Then $I_R \ll R_R$ or I_R is a direct summand of R_R. Applying Theorem 11, we obtain that $I_R \ll R_R$. Thus Soc $R_R \subseteq J(R)$.

(4) Since the class of singular modules is closed under sums, it is clear.

(5) By Theorem 11, all simple right R-modules are singular and so all maximal right ideals of R are essential in R_R. □

Let M be a module and U and V be submodules of M. The submodule V is said to be a *supplement* of U in M or U is said to have a *supplement* V in M if V is minimal with respect to $M = U + V$. It is well known that a submodule V of M is a supplement of U in M if and only if $M = U + V$ and $U \cap V \ll V$. M is called *supplemented* if every submodule of M has a supplement in M (see [15, 41]).

In [16], Zhou generalizes small submodules to δ-small submodules of a module M as follows. A submodule $N \subseteq M$ is said to be δ-*small* in M and indicated by $N \ll_\delta M$ if $N + K \neq M$ for every proper submodule K of M with M/K singular. It is clear that every small submodule or projective semisimple submodule of M is δ-small in M.

Let M be a module. In [12], M is said to be δ-*supplemented* if every submodule U of M has a δ-*supplement* V in M, that is, $M = U + V$ and $U \cap V \ll_\delta V$. Clearly supplemented modules are δ-supplemented. Furthermore, over a commutative domain, δ-supplemented modules are supplemented, but it is not generally true that every δ-supplemented module is supplemented.

Proposition 9 *Let R be a right SW-ring which is not a right V-ring and M be an R-module. Then, a submodule K of M is δ-small in M if and only if it is small in M. In particular, every δ-supplemented R-module is supplemented.*

Proof Let K be a δ-small submodule of M and $K + L = M$ hold for some submodule L of M. By [16, Lemma 1.2], there exists a projective semisimple submodule $K^{'}$ of K such that $M = K^{'} \oplus L$. Then $K^{'} = 0$ by Theorem 11 and so $L = M$. It means that K is a small submodule of M. □

Let M be a module. A projective cover of the module M is an epimorphism $f\colon P \longrightarrow M$ from a projective module P to M such that Ker f is a small submodule of P. It follows that if a module M has a projective cover, M has a maximal submodule (that is, Rad $M \neq M$). A ring R is called *a right perfect ring* if every right R-module has a projective cover ([15, 43.9]). It is known in [15, 43.9] that a ring R is right perfect if and only if every right R-module is supplemented.

Zhou [16] generalized right perfect rings to right δ-perfect rings as follows. A projective module P is called a *projective δ-cover* of a module M if there exists an epimorphism $f\colon P \longrightarrow M$ with Ker $f \ll_\delta P$. A ring R is called *right δ-perfect* if every right R-module has a projective δ-cover. It is proven in [12, Theorem 3.3 and Theorem 3.4] that a ring R is right δ-perfect if and only if every right R-module is δ-supplemented. Using these facts along with the Proposition above, we obtain the following result.

Corollary 10 *Let R be a right SW-ring which is not a right V-ring. Then, R is right δ-perfect if and only if it is right perfect.*

Following [1], a module M is called SCS if every closed simple submodule of M is a direct summand of M. Using Theorem 11, we have the following fact.

Proposition 10 *Let R be a right SW-ring which is not a right V-ring. If R_R is an SCS module, then there is no closed simple submodule of R_R.*

Proof Let S be a closed simple submodule of R_R. Then, S is a projective module as a direct summand of the module R_R. Since R is a right SW-ring which is not a right V-ring, by Theorem 11, S is singular. This is a contradiction. □

Since simple-destitute rings are right SW-rings, a commutative local ring is an SW-ring by [4, Lemma 4.6]. But a commutative SW-ring need not be local (see Example 2 (1)).

Theorem 12 *Let R be a commutative Noetherian ring. Then the following statements are equivalent.*

(1) *R is a simple-destitute ring.*
(2) *R is an SW-ring.*
(3) *R is either local or semisimple Artinian.*

Proof (1) $\Rightarrow$ (2) is clear and (3) $\Rightarrow$ (1) follows from [5, Theorem 5.5].

(2) $\Rightarrow$ (3) If R is not a local ring, then there are distinct maximal ideals T_1 and T_2 of R. Since R is an SW-ring, simple R-modules R/T_1 and R/T_2 are working-class. By [14, Proposition 4.21], we can write $\mathrm{Hom}(E(R/T_2), E(R/T_1)) = 0$ and so R/T_1 is $E(R/T_2)$-injective. Therefore $\mathrm{Rad}\, E(R/T_2) = 0$. It follows that $E(R/T_2) = R/T_2$, that is, R/T_2 is injective. Applying Theorem 1, the ring R is a V-ring. This completes the proof. □

Example 3 Let p be a prime integer and $n > 1$. By Theorem 12, the ring $\mathbb{Z}/p^n\mathbb{Z}$ is an SW-ring which is not a V-ring.

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