



On the existence of positive solutions for the second-order boundary value problem



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ABSTRACT

This paper is concerned with the existence of positive solutions to a second order boundary value problem. By imposing growth conditions on f and using a generalization of the Leggett–Williams fixed point theorem, we prove the existence of at least three symmetric positive solutions.

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1. Introduction

In the past 20 years, there has been attention focused on the existence of positive solutions to boundary value problems for ordinary differential equations; see [1–15]. It is well known that the Krasnosel'skii [16] fixed point theorems and the Leggett–Williams [17] multiple fixed-point theorem play an extremely important role.

In this paper, we discuss the existence of at least three positive solutions to the following boundary value problem:

$$u''(t) + f(u(t)) = 0, \quad t \in [0, 1], \quad (1.1)$$

$$u'(0) = 0, \quad u(1) = 0, \quad (1.2)$$

where $f : \mathbb{R} \rightarrow [0, \infty)$ is continuous. A solution $u \in C^{(2)}[0, 1]$ of (1.1), (1.2) is both nonnegative and concave on $[0, 1]$. We impose growth conditions on f which allows us to apply the generalization of the Leggett–Williams fixed point theorem in finding three symmetric positive solutions of (1.1), (1.2).

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2. Preliminaries

In this section, we give some background material concerning cone theory in a Banach space, and we then state the generalization of the Leggett–Williams fixed-point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:

- (i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
- (ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P in a real Banach space E if $\alpha : P \rightarrow [0, \infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P in a real Banach space E if $\beta : P \rightarrow [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative continuous convex functionals on P , and α, ψ be nonnegative continuous concave functionals on P . Then for nonnegative real numbers h, a, b, d and c , we define the following convex sets:

$$P(\gamma, c) = \{u \in P : \gamma(u) < c\},$$

$$P(\gamma, \alpha, a, c) = \{u \in P : a \leq \alpha(u), \gamma(u) \leq c\},$$

$$Q(\gamma, \beta, d, c) = \{u \in P : \beta(u) \leq d, \gamma(u) \leq c\},$$

$$P(\gamma, \theta, \alpha, a, b, c) = \{u \in P : a \leq \alpha(u), \theta(u) \leq b, \gamma(u) \leq c\},$$

$$Q(\gamma, \beta, \psi, h, d, c) = \{u \in P : h \leq \psi(u), \beta(u) \leq d, \gamma(u) \leq c\}.$$

We consider the two-point boundary value problem

$$-u'' = h(t), \quad t \in [0, 1], \quad (2.1)$$

$$u'(0) = 0, \quad u(1) = 0. \quad (2.2)$$

Lemma 2.1. Let $h \in L^1[0, 1]$. Then the two-point boundary value problem (2.1) and (2.2) has a unique solution

$$u(t) = \int_0^1 G(t, s)h(s)ds$$

where Green's function $G(t, s)$ is

$$G(t, s) = \begin{cases} 1-t, & 0 \leq s \leq t \leq 1, \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases}$$

The following is a generalization of the Leggett–Williams fixed-point theorem which will play an important role in the proof of our main results.

Theorem 2.1 ([18]). *Let P be a cone in a real Banach space E . Suppose there exist positive numbers c and M , nonnegative continuous concave functionals α and ψ on P , and nonnegative continuous convex functionals γ, β and θ on P with*

$$\alpha(u) \leq \beta(u), \quad \|u\| \leq M\gamma(u),$$

for all $u \in \overline{P(\gamma, c)}$. Suppose that $F : \overline{P(\gamma, c)} \rightarrow \overline{P(\gamma, c)}$ is a completely continuous operator and that there exist nonnegative numbers h, d, a, b , with $0 < d < a$ such that:

- (B1) $\{u \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(u) > a\} \neq \emptyset$ and $\alpha(Fu) > a$ for $u \in P(\gamma, \theta, \alpha, a, b, c)$;
- (B2) $\{u \in Q(\gamma, \beta, \psi, h, d, c) : \beta(u) < d\} \neq \emptyset$ and $\beta(Fu) < d$ for $u \in Q(\gamma, \beta, \psi, h, d, c)$;
- (B3) $\alpha(Fu) > a$, for $u \in P(\gamma, \alpha, a, c)$ with $\theta(Fu) > b$;
- (B4) $\beta(Fu) < d$, for $u \in Q(\gamma, \beta, d, c)$ with $\psi(Fu) < h$.

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(u_1) < d, \quad a < \alpha(u_2) \quad \text{and} \quad d < \beta(u_3), \quad \text{with} \quad \alpha(u_3) < a.$$

3. Main result

In this section, we impose the growth conditions on f which allow us to apply the generalization of the Leggett–Williams fixed-point theorem in establishing the existence of at least three positive solutions of (1.1) and (1.2). We will make use of various properties of Green’s function $G(t, s)$ which include

$$\begin{aligned} \int_0^1 G(t, s)ds &= \frac{1-t^2}{2}, \quad \text{for } 0 \leq t \leq 1, \\ \int_0^{1/r} G\left(\frac{1}{2}, s\right)ds &= \frac{1}{2r}, \quad \int_{1-(1/r)}^1 G\left(\frac{1}{2}, s\right)ds = \frac{1}{2r^2}, \quad \text{for } 2 < r, \\ \int_{1/r}^{1/2} G\left(\frac{1}{2}, s\right)ds &= \frac{r-2}{4r}, \quad \int_{1/2}^{1-(1/r)} G\left(\frac{1}{2}, s\right)ds = \frac{r^2-4}{8r^2}, \quad \text{for } 2 < r, \\ \int_{t_1}^{t_2} G(t_1, s)ds + \int_{1-t_2}^{1-t_1} G(t_1, s)ds &= t_2 - t_1, \quad \text{for } 0 < t_1 < t_2 \leq \frac{1}{2}, \\ \min_{r \in [0,1]} \frac{G(t_1, r)}{G(t_2, r)} &= 1, \quad \text{for } 0 < t_1 < t_2 \leq \frac{1}{2}, \quad \max_{r \in [0,1]} \frac{G(1/2, r)}{G(t, r)} = 1, \quad \text{for } 0 < t \leq \frac{1}{2}. \end{aligned}$$

Let $E = C[0, 1]$ be endowed with the maximum norm, $\|u\| = \max_{t \in [0,1]} |u(t)|$. Then for $0 < t_3 \leq 1/2$, we define the cone $P \subset E$ by

$$P = \{u \in E : u \text{ is concave, symmetric, nonnegative valued on } [0, 1], \min_{t \in [t_3, 1-t_3]} u(t) \geq 2t_3 \|u\|\}.$$

We define the nonnegative, continuous concave functionals α, ψ and nonnegative continuous convex functionals β, θ, γ on the cone P by

$$\begin{aligned} \alpha(u) &= \min_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} u(t) = u(t_1), \\ \beta(u) &= \max_{t \in [1/r, (r-1/r)]} u(t) = u\left(\frac{1}{2}\right), \\ \gamma(u) &= \max_{t \in [0, t_3] \cup [1-t_3, 1]} u(t) = u(t_3), \\ \theta(u) &= \max_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} u(t) = u(t_2), \\ \psi(u) &= \min_{t \in [1/r, (r-1/r)]} u(t) = u\left(\frac{1}{r}\right), \end{aligned}$$

where t_1, t_2 , and r are nonnegative numbers such that

$$0 < t_1 < t_2 \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{r} \leq t_2.$$

We see that, for all $u \in P$,

$$\alpha(u) = u(t_1) \leq u\left(\frac{1}{2}\right) = \beta(u), \quad (3.1)$$

$$\|u\| = u\left(\frac{1}{2}\right) \leq \frac{1}{2t_3}u(t_3) = \frac{1}{2t_3}\gamma(u), \quad (3.2)$$

and also that $u \in P$ is a solution of (1.1), (1.2) if and only if

$$u(t) = \int_0^1 G(t,s)f(u(s))ds, \quad \text{for } t \in [0, 1].$$

We now present our result of the paper.

Theorem 3.1. *Suppose that there exist nonnegative numbers a, b , and c such that $0 < a < b \leq \frac{ct_1}{t_2}$, and suppose that f satisfies the following growth conditions:*

(C1) $f(w) < (4r^2/(r^2 - 4))(a - (2c/(r(1 - t_3^2))))$, for $(2a/r) \leq w \leq a$;

(C2) $f(w) \geq b/(t_2 - t_1)$, for $b \leq w \leq (t_2b)/t_1$;

(C3) $f(w) \leq (2c)/(1 - t_3^2)$, for $0 \leq w \leq c/(2t_3)$.

Then the boundary value problem (1.1) and (1.2) has three symmetric positive solutions u_1, u_2 and u_3 satisfying

$$\begin{aligned} \max_{t \in [0, t_3] \cup [1-t_3, 1]} u_i(t) &\leq c, \quad \text{for } i = 1, 2, 3, \\ \min_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} u_1(t) &> b, \quad \max_{t \in [1/r, (r-1)/r]} u_2(t) < a, \\ \min_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} u_3(t) &< b, \quad \text{with } \max_{t \in [1/r, (r-1)/r]} u_3(t) > a. \end{aligned}$$

Proof. Let us define the completely continuous operator F by

$$(Fu)(t) = \int_0^1 G(t,s)f(u(s))ds.$$

We will seek fixed points of F in the cone. We note that, if $u \in P$, then from properties of $G(t, s)$, $Fu(t) \geq 0$, and $(Fu)''(t) = -f(u(t)) \leq 0$, $0 \leq t \leq 1$, $Fu(t_3) \geq 2t_3Fu(1/2)$, and $Fu(t) = Fu(1-t)$, $0 \leq t \leq 1/2$. This implies that $Fu \in P$, and so $F : P \rightarrow P$.

Now, for all $u \in P$, from (3.1), we get $\alpha(u) \leq \beta(u)$ and from (3.2), $\|u\| \leq \frac{1}{2t_3}\gamma(u)$.

If $u \in \overline{P(\gamma, c)}$, then $\|u\| \leq 1/(2t_3)\gamma(u) \leq c/(2t_3)$ and from (C3) we get,

$$\begin{aligned} \gamma(Fu) &= \max_{t \in [0, t_3] \cup [1-t_3, 1]} \int_0^1 G(t,s)f(u(s))ds \\ &= \int_0^1 G(t_3,s)f(u(s))ds \\ &\leq \left(\frac{2c}{1-t_3^2}\right) \int_0^1 G(t_3,s)ds = c. \end{aligned}$$

Thus, $F : \overline{P(\gamma, c)} \rightarrow \overline{P(\gamma, c)}$. It is immediate that

$$\left\{ u \in P\left(\gamma, \theta, \alpha, b, \frac{bt_2}{t_1}, c\right) : \alpha(u) > b \right\} \neq \emptyset \quad \text{and} \quad \left\{ u \in Q\left(\gamma, \beta, \psi, \frac{2a}{r}, a, c\right) : \beta(u) < a \right\} \neq \emptyset.$$

We will show the remaining conditions of [Theorem 2.1](#).

(1) If $u \in Q(\gamma, \beta, \psi, (2a)/r, a, c)$ with $\psi(Fu) < (2a)/r$ then $\beta(Fu) < a$.

$$\begin{aligned} \beta(Fu) &= \max_{t \in [1/r, (r-1)/r]} \int_0^1 G(t, s) f(u(s)) ds \\ &= \int_0^1 G\left(\frac{1}{2}, s\right) f(u(s)) ds \\ &= \int_0^1 \frac{G(1/2, s)}{G(1/r, s)} G\left(\frac{1}{r}, s\right) f(u(s)) ds \\ &\leq \int_0^1 G\left(\frac{1}{r}, s\right) f(u(s)) ds = \psi(Fu) < a. \end{aligned}$$

(2) If $u \in Q(\gamma, \beta, \psi, (2a)/r, a, c)$, then $\beta(Fu) < a$.

$$\begin{aligned} \beta(Fu) &= \max_{t \in [1/r, (r-1)/r]} \int_0^1 G(t, s) f(u(s)) ds \\ &= \int_0^1 G\left(\frac{1}{2}, s\right) f(u(s)) ds \\ &= 2 \int_0^{1/r} G\left(\frac{1}{2}, s\right) f(u(s)) ds + 2 \int_{1/r}^{1/2} G\left(\frac{1}{2}, s\right) f(u(s)) ds \\ &< \frac{2c}{r(1-t_3^2)} + \left(\frac{4r^2}{r^2-4}\right) \left(a - \frac{2c}{r(1-t_3^2)}\right) \left(\frac{r^2-4}{4r^2}\right) = a. \end{aligned}$$

(3) If $u \in Q(\gamma, \alpha, b, c)$ with $\theta(Fu) > (bt_2)/t_1$, then $\alpha(Fu) > b$.

$$\begin{aligned} \alpha(Fu) &= \min_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} \int_0^1 G(t, s) f(u(s)) ds \\ &= \int_0^1 G(t_1, s) f(u(s)) ds \\ &= \int_0^1 \frac{G(t_1, s)}{G(t_2, s)} G(t_2, s) f(u(s)) ds \\ &\geq \int_0^1 G(t_2, s) f(u(s)) ds = \theta(Fu) > b. \end{aligned}$$

(4) If $u \in Q(\gamma, \theta, \alpha, b, (bt_2)/t_1, c)$, then $\alpha(Fu) > b$.

$$\begin{aligned} \alpha(Fu) &= \min_{t \in [t_1, t_2] \cup [1-t_2, 1-t_1]} \int_0^1 G(t, s) f(u(s)) ds \\ &= \int_0^1 G(t_1, s) f(u(s)) ds \\ &> \int_{t_1}^{t_2} G(t_1, s) f(u(s)) ds + \int_{1-t_2}^{1-t_1} G(t_1, s) f(u(s)) ds \end{aligned}$$

$$\begin{aligned} &\geq \left(\frac{b}{t_2 - t_1}\right) \int_{t_1}^{t_2} G(t_1, s) ds + \left(\frac{b}{t_2 - t_1}\right) \int_{1-t_2}^{1-t_1} G(t_1, s) ds \\ &= \left(\frac{b}{t_2 - t_1}\right) \left(\frac{-2t_1 + t_1^2 + 2t_2 - t_2^2}{2}\right) + \left(\frac{b}{t_2 - t_1}\right) \left(\frac{-t_1^2 + t_2^2}{2}\right) = b. \end{aligned}$$

Since all the conditions of the generalized Leggett–Williams fixed point theorem are satisfied, (1.1), (1.2) has three positive solutions $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$\alpha(u_1) > b, \quad \beta(u_2) < a, \quad \alpha(u_3) < b, \quad \text{with } \beta(u_3) > a. \quad \square$$

Remark 3.1. When f is autonomous, we select to carry out the analysis. But, if $f = f(t, u(t))$ and moreover, for each fixed u , $f(t, u(t))$ is symmetric about $t = \frac{1}{2}$, then a similar theorem would be correct with respect to same cone P .

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