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Sergey Borisenok

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# Target Attractor Tracking of Relative Phase in Bosonic Josephson Junction

Sergey Borisenok

*Departmentof Electrical and Electronics Engineering,*
*Faculty of Engineering, Abdullah Gül University*
*Kocasinan - 38080 Kayseri, Turkey*

sergey.borisenok@agu.edu.tr

**Abstract.**The relative phase of Bosonic Josephson junction in the Josephson regime of Bose-Hubbard model is tracked via the target attractor ('synergetic') feedback algorithm with the inter-well coupling parameter presented as a control function. The efficiency of our approach is demonstrated numerically for Gaussian and harmonic types of target phases.

## INTRODUCTION

Being described as early as in 1962 [1], Josephson junction (JJ) plays an important role in the modern technology. For instance, they are used to detect magnetic fluxes in superconducting quantum interference devices (SQUIDs) for preparing and measuring the states of qubits at different stages of quantum computations [2]. Among various physical realizations the family of Bosonic JJs is in the focus of modern research due to the variety of experimental methods to design and control them.

As a particular case of Bosonic JJs, many publications deal with multi-atomic Bose-Einstein condensate (BEC) in an external field of coupled potentials. Modern experimental technique achieved great progress in non-demolition measurements of BEC systems [3,4]. The possibility of experimental laser driving of double-well Bosonic JJ has been demonstrated in [5]. It also can be controlled by a single trapped ion, like $^{171}Yb^{+}$ [6]. In [2] multi-JJ functions constructed with BEC in periodic potentials has been studied. Thus, this progress in experiment opened a gate for applying variety of control algorithms to manipulate with the parameters of Bosonic JJ. For the further details one can check the excellent review [7].

Mathematical models, together with Bose-Hubbard's among them, that represent different forms of Josephson junction, demonstrate the richness of their dynamical behavior. Numerically such systems have been studied intensively; see, for instance, the analysis of exact solutions to time-dependent multi-boson Schrödinger Eq. in [8]. Nevertheless, the set of proposed theoretical control algorithms for Bosonic JJs is relatively poor. In [9] the harmonic open-loop control (with Kapitza oscillator-type effect) has been applied for a small number of atoms $N = 100$.Feedback algorithms used in the JJ models are mostly basedon speed gradient (SG) [10,11] and target attractor feedback [12,13].

In this paper we develop our approach for feedback tracking dynamical properties of JJs. In[11] we demonstrated the efficiency of speed gradientalgorithm to control dynamical parameters of quantum (two-level) system in external field and in [14] we used SG to design a target current through superconducting Josephson junction; while in [15] we've manipulated with such JJ via Kolesnikov's target attractor ('synergetic' in the author terminology) algorithm by controlling signal external voltage. Here we transfer our method to a Bosonic realization of JJ, particularly to multi-atomic Bose-Einstein condensate (BEC) trapped in a double-well potential. We describe the features of its Bose-Hubbard model and apply Kolesnikov's algorithm to design a target relative phase. The role of control parameter now plays inter-well coupling rather than the inter-particle interaction parameter (the Bosonic analog of

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the external potential in our previous superconducting model) that is supposed to be fixed. The efficiency of our approach is demonstrated numerically for Gaussian and harmonic types of target phases.

## CONTROL ALGORITHM IN BOSE-HUBBARD MODEL

### Bose-Hubbard Model for Bosonic Josephson Junction

Let's consider $N$-atomic Bose-Einstein condensate trapped in a coupled potential of two wells marked with 1 and 2. Such system can berepresented with Bose-Hubbard (BH) dynamical model [7]:

$$\begin{aligned} \frac{dn}{dt} &= -E_j\sqrt{1-\frac{4n^2}{N^2}}\cdot\sin\varphi; \\ \frac{d\varphi}{dt} &= E_c n + E_j\cdot\frac{4n}{N^2}\left(1-\frac{4n^2}{N^2}\right)^{-1/2}\cos\varphi \end{aligned} \tag{1}$$

(the Planck constant is chosen to be 1). In (1) $E_j$ corresponds to the energy of the atomic tunneling through the potential inter-well barrier, $E_c$ stands for the energy of inter-particle interaction. The numbers of particles in the wells are $N_1$ and $N_2$, with the normalization for the total number of atoms:

$$N_1 + N_2 = N, \tag{2}$$

and the phase functions are $\varphi_1$ and $\varphi_2$. Two dynamical variables, theimbalance particle number and the relative phase, are given by:

$$n = \frac{N_1 - N_2}{2}\ ;\ \varphi = \varphi_1 - \varphi_2, \tag{3}$$

Now let's denote the fractional imbalance of the atomic population with:

$$z = \frac{2n}{N}, \tag{4}$$

and, correspondingly, the dimensionless inter-well coupling and interaction parameters with:

$$\omega = \frac{2E_j}{N}\ ;\ g = \frac{NE_c}{2}. \tag{5}$$

Often in the literature in the place of $\omega$ one can find $v = \omega/2$ called the effective tunneling strength.The ratio $r = \omega/g$ defines the type of regime in the model: how week is the inter-atom interaction in comparison to the effect of the particle tunneling, see Table 1.

After the substitution of (4) and (5) into (1) we get the dimensionless dynamical system in the form:

$$\frac{dz}{dt} = -\omega\sqrt{1-z^2}\cdot\sin\varphi; \tag{6a}$$

$$\frac{d\varphi}{dt} = z\cdot\left(g + \frac{\omega}{\sqrt{1-z^2}}\cos\varphi\right). \tag{6b}$$

**TABLE 1.** Three regimes in the BH model.

| $r = \omega/g$ | **Regime** | **Inter-Atom Interaction** |
|---|---|---|
| $r \gg 1$ | Rabi's | Weak |
| $1/N^2 \ll r \ll 1$ | Josephson's | Intermediate |
| $r \ll 1/N^2$ | Fock's | Strong |

In the case of constant parameters $\omega$ and $g$ the system (6) is conservative and can be expressedvia the dimensionless Hamiltonian:

$$H = -\omega\sqrt{1-z^2}\cos\varphi + g\frac{z^2}{2}. \tag{7}$$

However, in our model we control the relative phase $\varphi$ with an appropriate choice of time-dependent coupling parameter $\omega(t)$, while the interaction parameter $g$ is supposed to be fixed.

## Target Attractor Algorithm

The target attractor control, or 'synergetic control' in author's terminology, is defined through the directed self-organization of the dynamical system [12]. The subset referring the control target is represented with an $m$-parametric attracting invariant manifold:

$$\psi_s(x_1,...,x_n) = 0\ ;\ s = 1...m, \tag{8}$$

with the functions of the state variables $x_1,\ldots,x_n$. Eqs (8) are constructed in the manner to provide the asymptotic stability of the system dynamics under constrain of the control target. For that the minimum of the following optimizing functional has to be satisfied:

$$J = \int_0^\infty \left( \sum_{s=1}^m \left[ T_s^2 \left( \frac{d\psi_s(t)}{dt} \right)^2 + \psi_s^2(t) \right] \right) dt = \min\ , \tag{9}$$

where $T_s$ are time scales (positive constants). To achieve the minimum (9) in exponent asymptotic, one can define the feedback as a set of $m$ equations for the following observers [13]:

$$T_s\frac{d\psi_s(t)}{dt} + \psi_s(t) = 0\,. \tag{10}$$

Tending to zeros, the observers (10) together with (8) lead the dynamical evolution of the system to the target attractor.

In our model, for the purpose of tracking (designing) the relative phase $\varphi$ with the target function $\varphi_*(t)$, we re-write (10) in the form:

$$\frac{d}{dt}\left[\varphi(t) - \varphi_*(t)\right] = -\frac{1}{T}\left[\varphi(t) - \varphi_*(t)\right]. \tag{11}$$

Eq.(11) converges to the target relative phase exponentially. The algorithm (11) may be interpreted as a Pecora-Carroll synchronization. Indeed, the relative phase $\varphi(t)$ works as a synchronizing link with the ideal (target) dynamical system, while the slave system (11) and (6b) contains the partial observer for the master $\varphi_*(t)$ .

The main point of the algorithm is to restore the required control signal $\omega(t)$. It can be derived from the substitution of (11) into (6b). That implies for the control parameter $\omega$:

$$\omega = \frac{\sqrt{1-z^2}}{\cos\varphi}\left\{\frac{1}{z}\left[\frac{d\varphi_*}{dt} - \frac{1}{T}(\varphi - \varphi_*)\right] - g\right\}. \tag{12}$$

In the asymptotics (11), where $\varphi$ can be replaced with $\varphi_*$, we get by (12) and (6a):

$$\begin{aligned} \omega_*(t) &= \frac{\sqrt{1-z_*^2}}{\cos\varphi_*}\left[\frac{1}{z_*}\frac{d\varphi_*}{dt} - g\right]; \\ \frac{dz_*}{dt} &= -\omega_*(t)\sqrt{1-z_*^2}\,\sin\varphi_* \;, \end{aligned} \tag{13}$$

where the asymptotic control signal $\omega_*(t)$ is restored from the solution of the combined Eq.(13) for the asymptotic $z_*(t)$:

$$\frac{dz_*}{dt} = -\left(1-z_*^2\right)\cdot\left[\frac{1}{z_*}\frac{d\varphi_*}{dt} - g\right]\tan\varphi_*. \tag{14}$$

Eq.(14) could be hardly investigated analytically, but we are able to make a slight simplification.

Let's re-write (13)-(14) in so-called spinor representation:

$$z(t) = \cos\alpha(t)\;. \tag{15}$$

Then (13)-(14) are transformed into:

$$\frac{d\alpha_*}{dt} = \omega_*\sin\varphi_* \tag{16}$$

and

$$\frac{d\alpha_*}{dt} = \left[\tan\alpha_*\frac{d\varphi_*}{dt} - g\cdot\sin\alpha_*\right]\tan\varphi_*. \tag{17}$$

In principle, the interaction parameter $g$ in Josephson regime (see Table 1) is supposed to be large, but the factor in front of the target phase derivative $d\varphi_*/dt$ in (14) and (17) is not limited. That may cause the divergence of $\omega(t)$. To avoid it and to keep the validity of our BH model under the Josephson regime we should replace the coupling function with the correctedcontrol signal:

$$\omega_{[\mathrm{control}]} = F\left(\omega_*\right), \tag{18}$$

with the initial ratio $r(0) = \omega(0)/g \ll 1$ and the filtering function:

$$F(x) = \frac{1}{2}\left[(1+x)\,\mathrm{sign}\,(1+x) + (1-x)\,\mathrm{sign}\,(x-1)\right] = \begin{cases} -1, & x < -1; \\ x, & |x| \le 1; \\ 1, & x > 1. \end{cases} \tag{19}$$

Thus, the final scenario of our algorithm includes three stages:

I. Calculation of the initial form of the control signal $\omega_*(t)$ using (13) or equivalent Eqs (16)-(17);
II. Correction of the control signal with (18)-(19);
III. Application of the calculated $\omega_{[\mathrm{control}]}(t)$ to the dynamical system (6).

## NUMERICAL SIMULATIONS

To demonstrate the efficiency of our tracking algorithm we chose here two types of target function, one consists of Gaussian components, another is a superposition of harmonic functions.The number of atoms is taken: $N = 1000$. The typical meaning for model parameters are $g = 100$ and $\omega(0) = 10$, that corresponds to the Josephson regime with $r(0) = 0.1$. The control parameter $T = 0.1$; the initial conditions are: $z(0) = 1.0$ and $\varphi(0) = 0.1$.

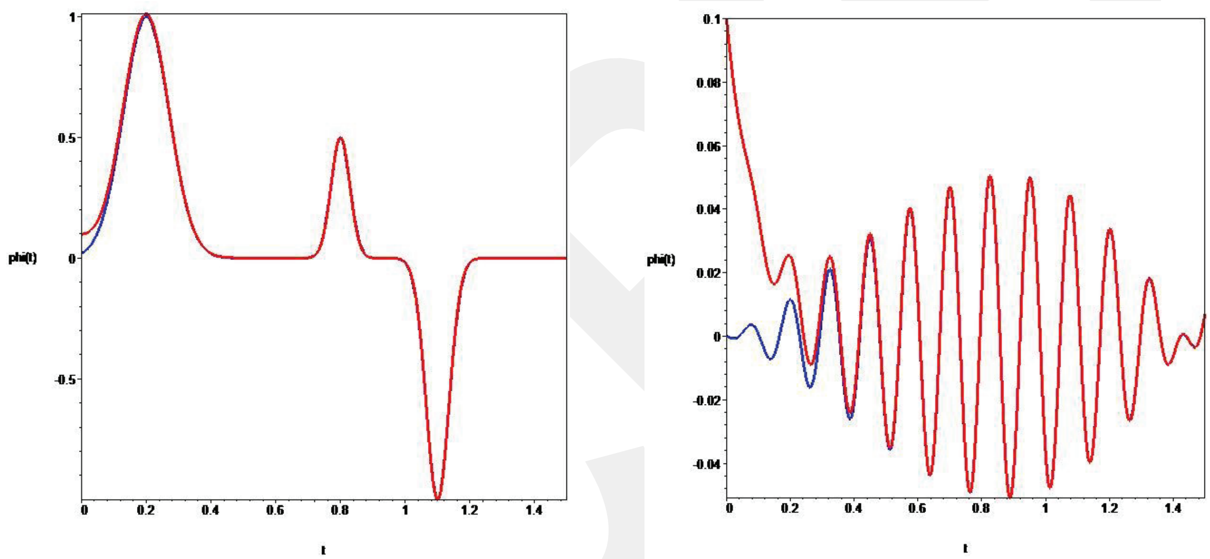
We test here two types of the target relative phase, the first one consists of three Gaussian components:

$$\varphi_*^{[\text{Gauss}]} = \exp\left\{-100\,(t-0.2)^2\right\} + \frac{1}{2}\exp\left\{-600\,(t-0.8)^2\right\} - \exp\left\{-400\,(t-1.1)^2\right\}, \tag{20}$$

and the second is a superposition of harmonic functions:

$$\varphi_*^{[\text{harm}]} = \sin(0.07t)\cdot\cos(50\,(t-0.7))\cdot\sin(\sqrt{3}(t+4)) \quad . \tag{21}$$

The numerical simulations for the phase tracking are plotted on Figs.1 and 2.

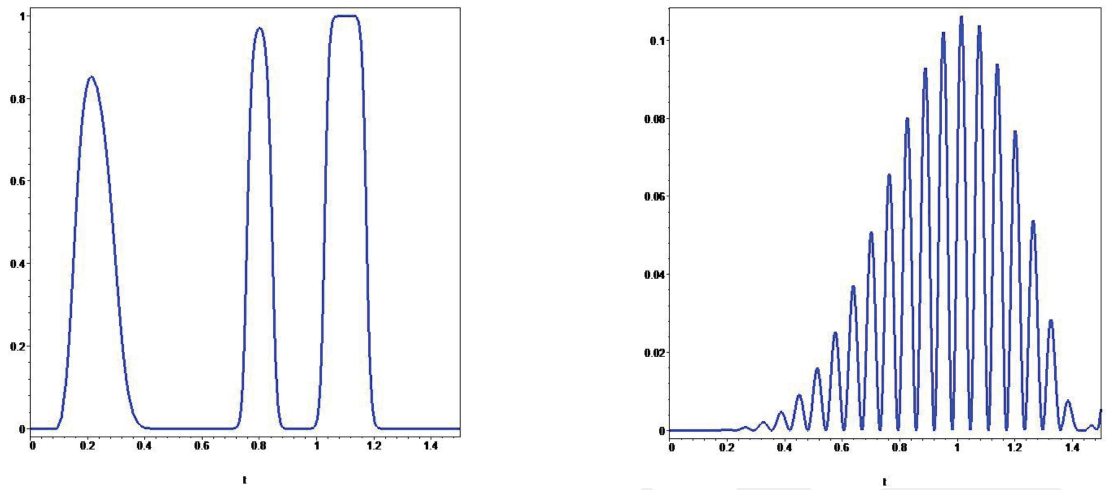



(a) (b)

**FIGURE 1.** Tracking of a given target function for the relative phase with 'synergetic' feedback algorithm. The target function is marked with blue line, the actual dynamical phase – with red line. (a) The Gaussian case (20). (b) The harmonic case (21).

On Fig.1 one can see the perfect matching for tracking goal (after $t = 0.2$ for the Gaussian case and after $t = 0.4$ for harmonic case).

The absolute value of corrected normalized control signal $r(t) = \omega_{[\text{control}]}(t)/g$ corresponding to the tracking on Fig.1. is shown on Fig.2.The plots on Fig.2 demonstrate that our numerical simulations satisfy the validity of BH model and, according to Table 1, stay mostly under the application of Josephson regime.




(a) (b)

**FIGURE 2.** The absolute value of corrected normalized control signal $r(t) = \omega_{[control]}(t)/g$ corresponding to the tracking on Fig.1. (a) The Gaussian case (20). (b) The harmonic case (21).

## CONCLUSIONS

The target attractor feedback applied to different realizations of Josephson junction, superconducting JJ, like in [15], or Bosonic JJ, like in this paper, with alternative definitions of the control signal,provides the efficient (with exponential asymptotic) tracking of dynamical characteristics of a given quantum system.

Nevertheless, one should consider that the algorithm drives the dynamical system with an extra strong external force to keep its evolution to be closed to the target attractor manifold.

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## REFERENCES

1. B. D. Josephson, Physics Letters **1**, 251 (1962).
2. Yu. Dumin, New Journal of Physics**11**, 103032 (2009).
3. M. Albiez, R. Gati, *et al*., Phys. Rev. Lett. **95**, 010402 (2005).
4. M. Zuppardo, J. P. Santos, *et al.*, Phys. Rev. A **91**, 033631 (2015)
5. G. Szirmai, G. Mazzarella, and L. Salasnich, Phys. Rev. A **91**, 023601 (2015).
6. R. Gerritsma, A. Negretti, *et al*., Phys. Rev. Lett. **109**, 080402 (2012).
7. R. Gati and M. K. Oberthaler, J. Phys. B: At. Mol. Opt. Phys. **40**, R61-R89 (2007).
8. K. Sakmann, A. I. Streltsov, *et al*., Phys. Rev. Lett. **103**, 220601 (2009).
9. E. Boukobza, M. G. Moore, *et al.*, Phys. Rev. Lett. **104**, 240402 (2010).
10. A. Fradkov, *Cybernetical Physics: From Control of Chaos to Quantum Control* (Springer, Berlin, 2007).
11. S. Borisenok, A. Fradkov, A. Proskurnikov, Periodic Control Systems **4(1)**, 81-86 (2010).
12. A. Kolesnikov, *Synergetic Control Methods for Complex Systems* (URSS Publ., Moscow, 2012), pp. 123-125.
13. A. Kolesnikov, “Introduction of Synergetic Control”, in *2014 American Control Conference* Proceedings (Portland, 2014), pp. 3013-3016.
14. S. Borisenok, K. Güven, Ö. Müstecaplıoğlu, “Controlling tunneling current in superconducting Josephson junction via speed gradient tracking”, submitted (2016).
15. S. Borisenok, “Tracking with target attractor feedback in superconducting Josephson junction”, *7th International Conference on Physics and Control (PhysCon 2015)*, Books of Abstracts (Istanbul, 2015), p.65.