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On the existence of positive solutions of the *p*-Laplacian dynamic equations on time scales

Abdulkadir Dogan[*†]

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In this paper, we investigate the existence of positive solutions for a nonlinear *m*-point boundary value problem for the *p*-Laplacian dynamic equations on time scales, by applying a Krasnosel'skii's fixed point theorem. As an application, an example is included to demonstrate the main results. Copyright © 2017 John Wiley & Sons, Ltd.

Keywords: time scales; boundary value problem; *p*-Laplacian; positive solutions; fixed point theorem

1. Introduction

The theory of dynamic equation on time scales was initiated by Stefan Hilger in his PhD thesis in 1988 [1] as a means of a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. In recent years, it has received a considerable amount of interest and attracted the attention of many researchers. It is still a new area, and research in this area is rapidly growing. The study of time scales has led to several important applications, for example, in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models [2–5]. Some basic definitions on dynamical systems on time scales can be found in [2,6].

In [7], Ma, Du, and Ge considered the existence of monotone positive solutions for the following multipoint boundary value problem (BVP) with *p*-Laplacian operator,

$$(\phi_p(u'(t)))' + q(t)f(t,u(t)) = 0, \qquad t \in (0,1),$$
$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i).$$

The main tool is the monotone iterative technique.

In [8,9], Wang and Ge studied the equation

$$(\phi_p(u'(t)))' + a(t)f(t,u(t)) = 0, \qquad t \in (0,1),$$

with the following boundary conditions:

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$
$$u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).$$

By applying fixed point index theory, they found sufficient conditions for the existence of twin positive solutions for the aforementioned problems by constructing available operators.

In [10], He and Jiang studied the existence of positive solutions of the *p*-Laplacian dynamic equations on time scale

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + g(t)f(u(t)) = 0, \qquad t \in [0,T]_{\mathbb{T}},$$

Department of Applied Mathematics, Faculty of Computer Sciences, Abdullah Gul University, Kayseri 38039, Turkey
[*] *Correspondence to: Abdulkadir Dogan, Department of Applied Mathematics, Faculty of Computer Sciences, Abdullah Gul University, Kayseri 38039, Turkey.*
[†] *E-mail: abdulkadir.dogan@agu.edu.tr*

satisfying the boundary conditions

$$u(0) - B_0(u^{\Delta}(0)) = 0, \qquad u^{\Delta}(T) = 0,$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) + B_1(u^{\Delta}(T)) = 0,$$

where $\phi_p(s)$ is p-Laplacian operator. By defining an appropriate Banach space and cones, they imposed the growth conditions on f that allow them to apply the triple fixed point theorem in obtaining existence of at least three positive solutions.

Recently, there is much current interest in questions of positive solutions for three-point BVPs on time scales, one may see [11–20] and the references therein. But there have been few papers considering the p-Laplacian problems on time scales [21–23]. In this paper, we study the existence of at least three positive solutions to the following p-Laplacian multipoint BVP on time scales

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t,u(t)) = 0, \qquad t \in (0,T)_{\mathbb{T}}, \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta}(\xi_i)), \tag{1.2}$$

where $\phi_p(u)$ is p-Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u, \quad$ for $p > 1$, with $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\xi_i \in \mathbb{T}$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, and a_i, b_i, a, f satisfy

(H1) $a_i, b_i \in [0,+\infty)$, satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$;
(H2) $a(t) \in C_{ld}([0,T]_{\mathbb{T}}, [0,+\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;
(H3) $f \in C([0,T] \times [0,+\infty), [0,+\infty))$.

In this paper, we will establish two new theorems concerning two positive solutions of (1.1) and (1.2). Our work concentrates on the case when the nonlinear term does not satisfy the conditions of Theorem 3.1 in [24]. Our results generalize Theorem 3.1 in [24] and Theorem 3.1 in [9].

The conditions we used in the paper are different from those in [9,19,24]. The results in the paper improve those presented in [23].The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting.

The rest of this paper is arranged as follows: we state some basic time scale definitions and prove several preliminary results in Section 2. We state and prove two theorems for at least two positive solutions of (1.1) and (1.2) in Section 3. The existence of single positive solution of problems (1.1) and (1.2) is given in Section 4. In Section 5, we give an example to demonstrate our results.

2. Preliminary lemmas

Lemma 2.1
If $1 - \sum_{i=1}^{m-2} a_i \neq 0$ and $1 - \sum_{i=1}^{m-2} b_i \neq 0$, then for $h \in C_{ld}[0,T]_{\mathbb{T}}$,

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + h(t) = 0, \qquad t \in (0,T)_{\mathbb{T}}, \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta}(\xi_i)) \tag{2.2}$$

has the unique solution

$$u(t) = \int_0^t \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right)\Delta s + C_2, \tag{2.3}$$

where

$$C_1 = \frac{\int_0^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i},$$
$$C_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Proof
Let u be as in (2.3). Taking the delta derivative of (2.3), we have

$$u^{\Delta}(t) = \phi_q\left(C_1 - \int_0^t h(\tau)\nabla\tau\right);$$

moreover, we obtain

$$\phi_p(u^{\Delta}(t)) = C_1 - \int_0^t h(\tau)\nabla\tau,$$

taking the nabla derivative of this expression yields

$$(\phi_p(u^{\Delta}))^{\nabla} = -h(t).$$

Now, we prove that u satisfies the boundary conditions (BCs) in (2.2). On the one hand, we have

$$u(0) = C_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i},$$

and

$$u(\xi_i) = \int_0^{\xi_i} \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right)\Delta s + C_2$$

so that

$$\sum_{i=1}^{m-2} a_i u(\xi_i) = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i};$$

thus,

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i).$$

On the other hand, we can find

$$\phi_p(u^{\Delta}(T)) = C_1 - \int_0^T h(\tau)\nabla\tau = \frac{\sum_{i=1}^{m-2} b_i \int_0^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i},$$

and

$$\phi_p(u^{\Delta}(\xi_i)) = \frac{\int_0^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} - \int_0^{\xi_i} h(\tau)\nabla\tau = \frac{\int_0^T h(\tau)\nabla\tau - \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i},$$

so that

$$\sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta}(\xi_i)) = \frac{\sum_{i=1}^{m-2} b_i \int_0^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}.$$

Therefore,

$$\phi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta}(\xi_i)),$$

so that u given in (2.3) is a solution of (2.1) and (2.2).

It is easy to see that BVP $(\phi_p(u^{\Delta}))^{\nabla} = 0, u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, $\phi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^{\Delta}(\xi_i))$ has only trivial solution. Thus, u in (2.3) is the unique solution of (2.1) and (2.2). □

Lemma 2.2

Suppose (H1) holds for $h \in C_{ld}[0,T]_{\mathbb{T}}$ and $h(t) \geq 0$. Then, the unique solution u of (2.1) and (2.2) satisfies $u(t) \geq 0$, for $t \in [0,T]_{\mathbb{T}}$.

Proof

Let

$$\varphi_0(s) = \phi_q\left(C_1 - \int_0^s h(\tau)\nabla\tau\right).$$

Then, we have

$$C_1 - \int_0^s h(\tau)\nabla\tau = \frac{\int_s^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_s^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}.$$

If $s < \xi_i \quad \forall i$, then $\int_s^{\xi_i} h(\tau)\nabla\tau \leq \int_s^T h(\tau)\nabla\tau$, we have

$$\begin{aligned}
&\int_s^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_s^{\xi_i} h(\tau)\nabla\tau \\
&\geq \int_s^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_s^T h(\tau)\nabla\tau \\
&= \left(1 - \sum_{i=1}^{m-2} b_i\right) \int_s^T h(\tau)\nabla\tau \geq 0.
\end{aligned}$$

Because

$$\frac{\int_s^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_s^{\xi_i} h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \geq 0,$$

it follows that $\varphi_0(s) \geq 0$. According to Lemma 2.1, we obtain

$$u(0) = C_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0$$

and

$$u(T) = \int_0^T \varphi_0(s)\Delta s + C_2 = \int_0^T \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_0(s)\Delta s \geq 0.$$

If $t \in (0,T)_{\mathbb{T}}$, we have

$$u(t) = \int_0^t \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_0(s)\Delta s \geq 0.$$

Therefore, $u(t) \geq 0, \quad t \in [0,T]_{\mathbb{T}}$. □

Lemma 2.3

Suppose (H1) holds, for $h \in C_{ld}[0,T]_{\mathbb{T}}$ and $h(t) \geq 0$. Then, the unique solution u of (2.1) and (2.2) satisfies

$$\inf_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i\xi_i}{T-\sum_{i=1}^{m-2} a_i(T-\xi_i)}, \qquad \|u\| = \max_{t\in[0,T]_{\mathbb{T}}} |u(t)|.$$

Proof

Clearly, $u^{\Delta}(t) = \varphi_0(t) \geq 0$. This implies that

$$\min_{t\in[0,T]_{\mathbb{T}}} u(t) = u(0), \qquad \|u\| = u(T).$$

It is easy to see that $u^{\Delta}(t_2) \leq u^{\Delta}(t_1)$ for any $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Hence, u^{Δ} is a decreasing function on $[0,T]_{\mathbb{T}}$. This means that the graph of u is concave down on $(0,T)_{\mathbb{T}}$. For each $i \in \{1,2,\ldots,m-2\}$, we have

$$\frac{u(T)-u(0)}{T-0} \geq \frac{u(T)-u(\xi_i)}{T-\xi_i},$$

that is,

$$Tu(\xi_i) - \xi_i u(T) \geq (T-\xi_i)u(0),$$

so that

$$T\sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i\xi_i u(T) \geq \sum_{i=1}^{m-2} a_i(T-\xi_i)u(0).$$

With the boundary condition $u(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i)$, we obtain

$$u(0) \geq \frac{\sum_{i=1}^{m-2} a_i\xi_i}{T-\sum_{i=1}^{m-2} a_i(T-\xi_i)} u(T).$$

This completes the proof. ☐

Let the norm on $C_{ld}[0, T]$ be the maximum norm. Then, the $C_{ld}[0, T]$ is a Banach space. It is easy to see that BVPs (1.1) and (1.2) have a solution $u = u(t)$ if and only if the operator $u \to Su$

$$(Su)(t) = \int_0^t \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau\right)\Delta s + \tilde{C}_2,$$

where

$$\begin{aligned}
\tilde{C}_1 &= \frac{\int_0^T a(\tau)f(\tau, u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i},\\
\tilde{C}_2 &= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i}.
\end{aligned}$$

Define the cone $P \subset E$ by

$$P = \left\{u : u \in C_{ld}[0, T],\ u(t) \geq 0,\ \inf_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma\|u\|\right\},$$

where γ is the same as in Lemma 2.3. It is obvious that P is a cone in $C_{ld}[0, T]$.

Lemma 2.4
$S : P \to P$ is completely continuous.

Proof
Firstly, we show that $S(P) \subset P$.

$\forall u \in P$, it is easy to check that Su is nonnegative, concave, and decreasing on $[0, T]_{\mathbb{T}}$. Thus, $Su \in P$. Moreover, we know that Su satisfies (2.2). Hence, Lemma 2.3 implies the Harnack-like inequality

$$\inf_{t\in[0,T]_{\mathbb{T}}} (Su)(t) \geq \gamma\|Su\|, \text{ for } u \in P,$$

that is, $Su \in P$. Therefore, we can find that $S(P) \subset P$.

Secondly, we show that S maps bounded set into itself. Suppose that $c > 0$ is a constant and $u \in \overline{P_c} = \left\{u \in P : \|u\| \leq c\right\}$. Note that the continuity of f guarantees that there is a $L > 0$ such that $f(t, u(t)) \leq \phi_p(L)$ for $t \in [0, T]_{\mathbb{T}}$. Therefore,

$$\begin{aligned}
\|Su\| &= \max_{0\leq t\leq T} (Su)(t)\\
&= \int_0^T \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau\right)\Delta s\\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau\right)\Delta s\\
&= \int_0^T \phi_q\left(\frac{\int_0^T a(\tau)f(\tau, u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\frac{\int_0^T a(\tau)f(\tau, u(\tau))\nabla\tau - \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&= \int_0^T \phi_q\left(\frac{\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^0 a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right.\\
&+\left.\frac{\sum_{i=1}^{m-2}b_i\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\Delta s\\
\le&\int_0^T\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right.\\
&+\left.\frac{\sum_{i=1}^{m-2}b_i\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\Delta s\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right.\\
&+\left.\frac{\sum_{i=1}^{m-2}b_i\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\Delta s\\
=&\int_0^T\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\Delta s\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right.\\
&+\left.\frac{\sum_{i=1}^{m-2}b_i\int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\Delta s\\
\le&L\phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right)\times\left(\int_0^T\Delta s+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\Delta s}{1-\sum_{i=1}^{m-2}a_i}\right).
\end{aligned}
$$

That is, $\overline{SP_c}$ is uniformly bounded.

Thirdly, for $t_1,t_2\in[0,T]_{\mathbb{T}}$, we have

$$
\begin{aligned}
\Big|(Su)(t_1)-(Su)(t_2)\Big|&=\left|\int_{t_1}^{t_2}\phi_q\left(\tilde{C}_1-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\right|\\
&\le L|t_1-t_2|\phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2}b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\right).
\end{aligned}
$$

Therefore, by applying the Arzela–Ascoli theorem on time scales [25], we can find that $\overline{SP_c}$ is relatively compact.

Finally, from the continuity of f and $a(t)\in C_{ld}([0,T]_{\mathbb{T}},[0,+\infty))$, we can find that S is continuous. Thus, S is completely continuous. This proof is complete. □

Lemma 2.5
Let

$$\varphi(s)=\phi_q\left(\tilde{C}_1-\int_0^s a(\tau]f(\tau,u(\tau))\nabla\tau\right).$$

For ξ_i $(i=1,\dots,m-2)$, then $\varphi(s)\ge 0$.

Proof
Because

$$\varphi(s)=\phi_q\left(\tilde{C}_1-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right),$$

then we have

$$
\begin{aligned}
&\tilde{C}_1-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\\
&=\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}\\
&\quad-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\\
&=\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau+\int_s^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}b_i}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
=&\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&-\left(\frac{\sum_{i=1}^{m-2} b_i \int_s^0 a(\tau)f(\tau,u(\tau))\nabla\tau+\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\\
=&\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2} b_i \int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}.
\end{aligned}
$$

If $s<\xi_i \quad \forall i$, then $\quad \int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau \le \int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau$, we have

$$
\begin{aligned}
&\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2} b_i \int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau\\
\ge&\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2} b_i \int_s^{T} a(\tau)f(\tau,u(\tau))\nabla\tau\\
=&\left(1-\sum_{i=1}^{m-2} b_i\right)\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau\ge 0.
\end{aligned}
$$

Because

$$
\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\sum_{i=1}^{m-2} b_i \int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\ge 0,
$$

then $\varphi(s)\ge 0$. This proof is complete. □

The proof of our main result is based upon an application of the following fixed point theorem in a cone.

Theorem 2.1 ([26])
Let E be a Banach space, and let $P\subset E$ be a cone. Suppose Ω_1,Ω_2 are open bounded subsets of E with $0\in\Omega_1,\bar{\Omega}_1\subset\Omega_2$, and let $S: P\cap(\bar{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator such that

(a) $\|Su\|\le\|u\|,\quad u\in P\cap\partial\Omega_1,$ and $\|Su\|\ge\|u\|,\ u\in P\cap\partial\Omega_2$; or
(b) $\|Su\|\ge\|u\|,\quad u\in P\cap\partial\Omega_1,$ and $\|Su\|\le\|u\|,\ u\in P\cap\partial\Omega_2$.

Then, S has at least one fixed point in $P\cap(\bar{\Omega}_2\backslash\Omega_1)$.

3. Two positive solutions

For $l>0,\quad \Omega_l=\{u\in P:\|u\|<l\},\quad \partial\Omega_l=\{u\in P:\|u\|=l\}$,

$$
\alpha(l)=\sup\{\|Su\|:u\in\partial\Omega_l\},\qquad \beta(l)=\inf\{\|Su\|:u\in\partial\Omega_l\},
$$

by Lemma 2.2, α and β are well defined.

Throughout this paper, we will suppose that $0\le\mu\le\nu\le T$. Now, for convenience, we introduce the following notation. Let

$$
\begin{aligned}
M_1=&\left\{\int_0^T \phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\right.\\
&\left.+\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^T \phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\right\}^{-1},\\
M_2=&\left\{\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)\nabla\tau\right)\Delta s\right\}^{-1}.
\end{aligned}
$$

Theorem 3.1
Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H4) $\quad p_i\in C([0,+\infty),[0,+\infty)),\ \ i=1,2$ and

$$
\overline{\lim_{l\to 0}}\frac{p_1(l)}{l^{p-1}}<M_1^{p-1},\qquad \overline{\lim_{l\to\infty}}\frac{p_2(l)}{l^{p-1}}<M_1^{p-1};
$$

(H5) $k_i \in ([0,T],[0,+\infty)),\ \ i=1,2;$

(H6) There exist $0 < c_1 \le c_2,\ \ 0 \le \lambda_2 < p-1 < \lambda_1$, such that

$$\begin{aligned} f(t,l) &\le p_1(l) + k_1(t) l^{\lambda_1}, && (t,l) \in [0,T]\times[0,c_1],\\ f(t,l) &\le p_2(l) + k_2(t) l^{\lambda_2}, && (t,l) \in [0,T]\times[c_2,+\infty); \end{aligned}$$

(H7) There exist $b > 0$, such that

$$\min\{f(t,l) : (t,l) \in [\mu,\nu]_{\mathbb{T}} \times [\gamma b, b]\} \ge (bM_2)^{p-1},$$

where μ, ν are nonnegative constants and $\mu, \nu \in \mathbb{T}$.

Then, BVPs (1.1) and (1.2) have at least two positive solutions u_1^*, u_2^* satisfying $0 < \|u_1^*\| < b < \|u_2^*\|$.

Proof
Let

$$\epsilon = \frac{1}{2}\min\left\{M_1^{p-1} - \overline{\lim_{l\to 0}} \frac{p_1(l)}{l^{p-1}}, M_1^{p-1} - \overline{\lim_{l\to\infty}} \frac{p_2(l)}{l^{p-1}}\right\}.$$

Then, there exist $0 < \overline{a_1} \le c_1, c_2 < \overline{a_2} < +\infty$, such that

$$\begin{aligned} p_1(l) &\le (M_1^{p-1} - \epsilon) l^{p-1}, && 0 \le l \le \overline{a_1},\\ p_2(l) &\le (M_1^{p-1} - \epsilon) l^{p-1}, && \overline{a_2} \le l \le +\infty. \end{aligned}$$

If $0 < l \le \overline{a_1}, u \in \partial\Omega_l$, then, $0 \le u(t) \le l,\ 0 \le t \le T$. From assumption (H6), we obtain

$$\begin{aligned} f(t,u(t)) &\le p_1(u(t)) + k_1(t) u^{\lambda_1}(t)\\ &\le (M_1^{p-1} - \epsilon) u^{p-1}(t) + k_1(t) u^{\lambda_1}(t)\\ &\le (M_1^{p-1} - \epsilon)\|u\|^{p-1} + k_1(t)\|u\|^{\lambda_1}\\ &= (M_1^{p-1} - \epsilon) l^{p-1} + k_1(t) l^{\lambda_1}. \end{aligned}$$

So that

$$\begin{aligned} &\tilde{C}_1 - \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau\\ &= \frac{\int_0^T a(\tau) f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\quad - \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau\\ &= \frac{\int_0^T a(\tau) f(\tau,u(\tau))\nabla\tau + \int_s^0 a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &= \frac{\int_s^T a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\le \frac{\int_s^T a(\tau) f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\le \frac{\int_s^T a(\tau) f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\le \frac{\int_s^T a(\tau)\Big[(M_1^{p-1} - \epsilon) l^{p-1} + k_1(\tau) l^{\lambda_1}\Big]\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\\ &\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)\Big[(M_1^{p-1} - \epsilon) l^{p-1} + k_1(\tau) l^{\lambda_1}\Big]\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}. \end{aligned}$$

So

$$
\begin{aligned}
\|Su\| &= \max_{0\le t\le T}(Su)(t)\\
&= \int_0^T \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&= \int_0^T \phi_q\Bigg(\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&= \int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\le \int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^0 a(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&= \int_0^T \phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\le \int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)[(M_1^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)[(M_1^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)[(M_1^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)[(M_1^{p-1}-\epsilon)l^{p-1}+k_1(\tau)l^{\lambda_1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s.
\end{aligned}
$$

It follows that

$$\begin{aligned}\frac{\alpha(l)}{l} \leq \int_0^T \phi_q\Bigg(&\frac{\int_s^T a(\tau)[M_1^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\ &+\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)[M_1^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\ &+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)[M_1^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\ &+\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)[M_1^{p-1}-\epsilon+k_1(\tau)l^{\lambda_1-p+1}]\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s.\end{aligned}$$

Noticing $\lambda_1 - p + 1 > 0$, we obtain

$$\begin{aligned}\overline{\lim_{l\to 0^+}}\frac{\alpha(l)}{l} &\leq \int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)(M_1^{p-1}-\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\ &\quad+\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)(M_1^{p-1}-\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\ &\quad+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)(M_1^{p-1}-\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\ &\quad+\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)(M_1^{p-1}-\epsilon)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\\ &=\left(M_1^{p-1}-\epsilon\right)^{\frac{1}{p-1}}\Bigg\{\int_0^T \phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\\ &\quad+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)\nabla\tau+\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\Bigg\}\\ &=\left(M_1^{p-1}-\epsilon\right)^{\frac{1}{p-1}} M_1^{-1}\\ &=\left(1-M_1^{-(p-1)}\epsilon\right)^{\frac{1}{p-1}}<1.\end{aligned}$$

So, there exists $0 < a_1 < \overline{a_1}$, such that $\alpha(a_1) < a_1$. This implies that $\|Su\| < \|u\|,\ u \in \partial\Omega_{a_1}$.

If $\overline{a_2} \leq l < +\infty$ and $u \in \partial\Omega_l$, then $0 \leq u(t) \leq l$. Similar to the aforementioned argument, noticing that $\lambda_2 - p + 1 < 0$, we can obtain $\overline{\lim}_{l\to\infty}\frac{\alpha(l)}{l} < 1$. So there exists $0 < \overline{a_2} < a_2$, such that $\alpha(a_2) < a_2$. It implies that $\|Su\| < \|u\|,\ u \in \partial\Omega_{a_2}$.

On the other hand, because $f : [0,T] \times [0,+\infty) \to [0,+\infty)$ is continuous, from assumption (H7), there exist $a_1 < b_1 < b < b_2 < a_2$, such that

$$\min\{f(t,l) : (t,l) \in [\mu,\nu] \times [\gamma b, b]\} \geq (b_i M_2)^{p-1},\quad i = 1,2.$$

If $u \in \partial\Omega_{b_1}$, then $\gamma b_1 \leq u(t) \leq b_1,\ \mu \leq t \leq \nu$. It follows that

$$\begin{aligned}\|Su\| &= \max_{0\leq t\leq T}(Su)(t)\\ &=\int_0^T \phi_q\left(\tilde{C_1}-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &\quad+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\phi_q\left(\tilde{C_1}-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &\geq\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\phi_q\left(\tilde{C_1}-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\ &=\frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\phi_q\Bigg(\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau-\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sum_{i=1}^{m-2} b_i \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i \int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
&\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i \int_s^{T} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)(b_1M_2)^{p-1}\nabla\tau\right)\Delta s \\
&= \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} b_1M_2 \int_\mu^\nu \phi_q\left(\int_s^T a(\tau)\nabla\tau\right)\Delta s \\
&= b_1M_2M_2^{-1} = b_1 = \|u\|.
\end{aligned}
$$

Similarly, we can show that if $u \in \partial\Omega_{b_2}$, then $\|Su\| \geq \|u\|$.

Now, we study the operator S on $\bar{\Omega}_{b_1}\backslash\Omega_{a_1}$ and $\bar{\Omega}_{a_2}\backslash\Omega_{b_2}$, respectively. From Theorem 2.1, we claim that the operator S has two fixed point $u_1^*, u_2^* \in P$, such that $a_1 \leq \|u_1^*\| \leq b_1$, $b_2 \leq \|u_1^*\| \leq a_2$. Therefore, $u_i^*, i = 1,2$, are positive solutions of problems (1.1) and (1.2). □

Theorem 3.2

Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H8) $p_i \in C([0,+\infty),[0,+\infty))$, $i = 3,4$, and

$$\underline{\lim}_{l\to 0+}\frac{p_3(l)}{l^{p-1}} > \left(\frac{M_2}{\gamma}\right)^{p-1}, \qquad \underline{\lim}_{l\to\infty}\frac{p_4(l)}{l^{p-1}} > \left(\frac{M_2}{\gamma}\right)^{p-1};$$

(H9) $k_i \in L^1([\mu,\nu],[0,+\infty))$, $i = 3,4$;

(H10) There exist $0 < c_3 \leq c_4$, $0 \leq \lambda_4 < p-1 < \lambda_3$, such that

$$
\begin{aligned}
f(t,l) &\geq p_3(l) - k_3(t)l^{\lambda_3}, \qquad (t,l) \in [\mu,\nu]\times[0,c_3],\\
f(t,l) &\geq p_4(l) - k_4(t)l^{\lambda_4}, \qquad (t,l) \in [\mu,\nu]\times[c_4,+\infty);
\end{aligned}
$$

(H11) There exist $a > 0$, such that

$$\max\{f(t,l) : (t,l) \in [0,T]\times[0,a]\} \leq (aM_1)^{p-1}.$$

Then, BVPs (1.1) and (1.2) have at least two positive solutions u_3^*, u_4^* satisfying $0 < \|u_3^*\| < a < \|u_4^*\|$.

Proof

Let

$$\epsilon = \frac{1}{2}\min\left\{\underline{\lim}_{l\to 0+}\frac{p_3(l)}{l^{p-1}} - \left(\frac{M_2}{\gamma}\right)^{p-1}, \underline{\lim}_{l\to\infty}\frac{p_4(l)}{l^{p-1}} - \left(\frac{M_2}{\gamma}\right)^{p-1}\right\}.$$

Then there exist $0 < \overline{b_3} \leq c_3, c_4 < \overline{b_4} < +\infty$, such that

$$p_3(l) \geq \left[\left(\frac{M_2}{\gamma}\right)^{p-1} + \epsilon\right] l^{p-1}, \qquad 0 \leq l \leq \overline{b_3},$$
$$p_4(l) \geq \left[\left(\frac{M_2}{\gamma}\right)^{p-1} + \epsilon\right] l^{p-1}, \qquad \overline{b_4} \leq l \leq +\infty.$$

If $0 < l \leq \overline{b_3}, u \in \partial\Omega_l$ then $\gamma l \leq u(t) \leq l,\ \mu \leq t \leq \nu$. From assumption (H10), we obtain

$$\begin{aligned}
\|Su\| &= \max_{0\leq t\leq T}(Su)(t)\\
&= \int_0^T \phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\tilde{C}_1 - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\frac{\int_0^T a(\tau)f(\tau,u(\tau))\nabla\tau - \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} b_i\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \sum_{i=1}^{m-2} b_i\int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. - \frac{\sum_{i=1}^{m-2} b_i\int_s^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right.\\
&\quad \left. - \frac{\sum_{i=1}^{m-2} b_i\int_s^{T} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right)\Delta s\\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s\\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)[p_3(u(\tau)) - k_3(\tau)u^{\lambda_3}]\nabla\tau\right)\Delta s\\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)\left[\left(\left(\frac{M_2}{\gamma}\right)^{p-1} + \epsilon\right)\right.\right.\\
&\quad \left.\left. \times (\gamma l)^{p-1} - k_3(\tau)l^{\lambda_3}\right]\nabla\tau\right)\Delta s.
\end{aligned}$$

It follows that

$$\frac{\beta(l)}{l} \geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)\left[M_2^{p-1} + \gamma^{p-1}\epsilon - k_3(\tau)l^{\lambda_3-p+1}\right]\nabla\tau\right)\Delta s.$$

Noticing $\lambda_3 - p + 1 > 0$, we have

$$\begin{aligned}
\underline{\lim}_{l\to 0^+}\frac{\beta(l)}{l} &\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_\mu^\nu \phi_q\left(\int_s^T a(\tau)\left(M_2^{p-1} + \gamma^{p-1}\epsilon\right)\nabla\tau\right)\Delta s\\
&= \left(M_2^{p-1} + \gamma^{p-1}\epsilon\right)^{\frac{1}{p-1}} M_2^{-1}\\
&= \left(1 + \gamma^{p-1}M_2^{-(p-1)}\epsilon\right)^{\frac{1}{p-1}} > 1.
\end{aligned}$$

So there exists b_3 with $0 < b_3 < a$, such that $\beta(b_3) > b_3$. It implies that $\|Su\| > \|u\|$, for $u \in \partial\Omega_{b_3}$.

If $\overline{b_4} \le \gamma l < +\infty$ and $u \in \partial\Omega_l$, then $\overline{b_4} \le \gamma l \le u(t) \le l$, $\mu \le t \le \nu$. As in the aforementioned argument, noticing that $\lambda_4 - p + 1 < 0$, we can obtain $\underline{\lim}_{l \to +\infty} \frac{\beta(l)}{l} > 1$. So, there exists b_4 with $0 < b_4 < +\infty$, such that $\beta(b_4) > b_4$. It implies that $\|Su\| > \|u\|$, $u \in \partial\Omega_{b_4}$. From assumption (H11), there exist $b_3 < a_3 < a < a_4 < b_4$, such that

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a_i]\} \le (a_i M_1)^{p-1}, \ \ i = 3,4.$$

If $u \in \partial\Omega_{a_3}$, then $0 \le u(t) \le a_3$, $0 \le t \le T$, and $f(t,u(t)) \le (a_3 M_1)^{p-1}$. It follows that

$$\begin{aligned}
\|Su\| \le & \int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \\
& + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
& + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \\
& + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
\le & \, a_3 M_1 \Bigg\{\int_0^T \phi_q\Bigg(\frac{\int_s^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s \\
& + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Bigg(\frac{\int_s^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg)\Delta s\Bigg\} \\
= & \, a_3 = \|u\|.
\end{aligned}$$

In the same way, if $u \in \partial\Omega_{a_4}$, then $\|Su\| \le \|u\|$.

Now, we consider the operator S on $\bar{\Omega}_{a_3} \backslash \Omega_{b_3}$ and $\bar{\Omega}_{b_4} \backslash \Omega_{a_4}$, respectively. From Theorem 2.1, we claim that the operator S has two fixed points $u_3^*, u_4^* \in P$, such that $b_3 \le \|u_1^*\| \le a_3$, $a_4 \le \|u_1^*\| \le b_4$. So $u_i^*, i = 3,4$, are positive solutions of problems (1.1) and (1.2). ☐

4. One positive solution

If the conditions of Theorem 3.1 and 3.2 are weakened, we will obtain the existence of single positive solution of problems (1.1) and (1.2).

Corollary 4.1

Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H12) $p_1 \in C([0,+\infty),[0,+\infty))$, and $\overline{\lim}_{l \to 0^+} \frac{p_1(l)}{l^{p-1}} < M_1^{p-1}$;

(H13) $k_1 \in L^1([0,T],[0,+\infty))$;

(H14) There exist $0 < c_1, p-1 < \lambda_1$, such that

$$f(t,l) \le p_1(l) + k_1(t) l^{\lambda_1}, \qquad (t,l) \in [0,T] \times [0,c_1];$$

(H15) There exist $0 < b$, such that

$$\min\{f(t,l) : (t,l) \in [\mu,\nu] \times [\gamma b, b]\} \ge (bM_2)^{p-1}.$$

Then, BVPs (1.1) and (1.2) have at least one positive solution.

Corollary 4.2

Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H16) $p_2 \in C([0,+\infty),[0,+\infty))$, and $\overline{\lim}_{l \to \infty} \frac{p_2(l)}{l^{p-1}} < M_1^{p-1}$;

(H17) $k_2 \in L^1([0,T],[0,+\infty))$;

(H18) There exist $0 < c_2, p-1 > \lambda_2 \ge 0$, such that

$$f(t,l) \le p_2(l) + k_2(t) l^{\lambda_2}, \qquad (t,l) \in [0,T] \times [c_2,+\infty);$$

(H19) There exist $0 < b$, such that

$$\min\{f(t,l) : (t,l) \in [\mu,\nu] \times [\gamma b, b]\} \geq (bM_2)^{p-1}.$$

Then, BVPs (1.1) and (1.2) have at least one positive solution.

Corollary 4.3
Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H20) $p_3 \in C([0,+\infty),[0,+\infty))$, and $\overline{\lim}_{l\to 0^+} \frac{p_3(l)}{l^{p-1}} > \left(\frac{M_2}{\gamma}\right)^{p-1}$;
(H21) $k_3 \in L^1([\mu,\nu],[0,+\infty))$;
(H22) There exist $0 < c_3, p-1 < \lambda_3$, such that

$$f(t,l) \geq p_3(l) - k_3(t)l^{\lambda_3}, \qquad (t,l) \in [\mu,\nu] \times [0,c_3];$$

(H23) There exists $0 < a$, such that

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a]\} \leq (aM_1)^{p-1}.$$

Then, BVPs (1.1) and (1.2) have at least one positive solution.

Corollary 4.4
Suppose (H1)–(H3) hold, and suppose that the following conditions hold:

(H24) $p_4 \in C([0,+\infty),[0,+\infty))$, and $\underline{\lim}_{l\to \infty} \frac{p_4(l)}{l^{p-1}} > \left(\frac{M_2}{\gamma}\right)^{p-1}$;
(H25) $k_4 \in L^1([\mu,\nu],[0,+\infty))$;
(H26) There exist $0 < c_4, p-1 > \lambda_4 \geq 0$, such that

$$f(t,l) \geq p_4(l) - k_4(t)l^{\lambda_4}, \qquad (t,l) \in [\mu,\nu] \times [c_4,+\infty);$$

(H27) There exists $0 < a$, such that

$$\max\{f(t,l) : (t,l) \in [0,T] \times [0,a]\} \leq (aM_1)^{p-1}.$$

Then, BVPs (1.1) and (1.2) have at least one positive solution.
The proof of the aforementioned results is similar to Theorems 3.1 and 3.2, we omit it.

5. Application

In this section, we present an example to explain our result. Let $\mathbb{T} = \mathbb{R}$, $(0,T) = (0,1)$, $f(t,0) \equiv 0$. We consider the following BVP on time scales

$$(\phi_3(u'))' + f(t,u(t)) = 0, \qquad t \in (0,1), \tag{5.1}$$

$$u(0) = \frac{1}{2}u\left(\frac{1}{2}\right), \qquad \phi_3(u'(1)) = \frac{1}{2}\phi_3\left(u'\left(\frac{1}{2}\right)\right), \tag{5.2}$$

where

$$f(t,u) = \begin{cases} 223u^3 + \min\left\{\frac{1}{\sqrt{t(1-t)}}, \frac{2}{u}\right\}\sqrt{u^5}, & \text{for } (t,u) \in [0,1]\times[0,1], \\ 225, & \text{for } (t,u) \in [0,1]\times[1,3], \\ 74u + \frac{\sqrt{3}}{6}\min\left\{\frac{1}{\sqrt{t(1-t)}}, \frac{2u}{3}\right\}\sqrt{u^3}, & \text{for } (t,u) \in [0,1]\times[3,+\infty). \end{cases}$$

It is easy to see that $f : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous. In this case, $p = 3$, $a(t) \equiv 1$, $m = 3$, $a_1 = b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\nu = 1$. Thus, we have by calculating that

$$\gamma = \frac{a_1\xi_1}{1 - a_1(1-\xi_1)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{1 - \frac{1}{2}(1-\frac{1}{2})} = \frac{1}{3},$$

$$M_1 = \left[\int_0^1 \phi_q\left(\frac{\int_s^1 d\tau + \frac{1}{2}\int_{\frac{1}{2}}^1 d\tau}{1-\frac{1}{2}}\right)ds + \frac{\frac{1}{2}}{1-\frac{1}{2}}\int_0^{\frac{1}{2}} \phi_q\left(\frac{\int_s^1 d\tau + \frac{1}{2}\int_{\frac{1}{2}}^1 d\tau}{1-\frac{1}{2}}\right)ds\right]^{-1}$$

$$= \left[\frac{-1+5\sqrt{5}}{6\sqrt{2}} + \frac{1}{12}(-3\sqrt{6}+5\sqrt{10})\right]^{-1} \approx 0.524932,$$

$$M_2 = \left[\frac{\frac{1}{2}}{1-\frac{1}{2}}\int_{\frac{1}{2}}^1 \phi_q\left(\int_s^1 d\tau\right)ds\right]^{-1}$$

$$= \left[\int_{\frac{1}{2}}^1 \sqrt{1-s}ds\right]^{-1} \approx 4.24264 < 5.$$

Now, we choose $c_1 = 1$, $c_2 = 3$, $b = 3, \lambda_1 = \frac{5}{2}$, $\lambda_2 = \frac{3}{2}$, $p_1(u) = 223u^3$, $p_2(u) = 74u$, $k_1(t) = k_2(t) = \frac{1}{\sqrt{t(1-t)}}$. Then, it is easy to see that

$$f(t,u) \le p_1(u) + k_1(t)u^{\frac{5}{2}}, \qquad (t,u) \in [0,1] \times [0,1],$$

$$f(t,u) \le p_2(u) + k_2(t)u^{\frac{3}{2}}, \qquad (t,u) \in [0,1] \times [3,+\infty),$$

$$\overline{\lim_{u\to 0}}\frac{p_1(u)}{u^2} = \overline{\lim_{u\to 0}}\frac{223u^3}{u^2} = 0 < M_1^2 = \left[\frac{-1+5\sqrt{5}}{6\sqrt{2}} + \frac{1}{12}(-3\sqrt{6}+5\sqrt{10})\right]^{-2} \approx 0.275554,$$

$$\overline{\lim_{u\to \infty}}\frac{p_2(u)}{u^2} = \overline{\lim_{u\to \infty}}\frac{74u}{u^2} = 0 < M_1^2 = \left[\frac{-1+5\sqrt{5}}{6\sqrt{2}} + \frac{1}{12}(-3\sqrt{6}+5\sqrt{10})\right]^{-2} \approx 0.275554,$$

$$\min\left\{f(t,u) : (t,u) \in \left[\frac{1}{2},1\right] \times [1,3]\right\} = 225 > (bM_2)^2 \approx 162.$$

It follows that f satisfies the conditions (H4)–(H7) of Theorem 3.1. Then, problems (1.1) and (1.2) have at least two positive solutions.

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