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Research Article

Analysis of the motion of a rigid rod on a circular surface using interpolated variational iteration method

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ABSTRACT

In this paper, interpolated variational iteration method (IVIM) is applied to investigate the vibration period and steady-state response for the motion of rigid rod rocking back and forth on a circular surface without slipping. The problem can be considered as a strongly nonlinear oscillator. In this solution procedure, analytical variational iteration technique is utilized by evaluating the integrals numerically. The approximate analytical results produced by the presented method are compared with the other existing solutions available in the literature. The advantage of using numerical evaluation of integrals, the method becomes fast convergent and a highly accurate solution can be obtained within seconds. The authors believe that the presented technique has potentially wide application in the other nonlinear oscillation problems.

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INTRODUCTION

It is crucial to obtain accurately the motion parameters of a nonlinear oscillation such as stability, fluctuation, vibration period and dynamic response to improve its performance [1-3]. As a well-known example of nonlinear vibration of oscillation systems, the motion of a rigid rod on a circular surface can be modelled by nonlinear governing differential equation [4]. Wu et al. [5] proposed a second order differential equation with the complex nonlinearities to obtain the vibration period and dynamic response of the rigid rod system.

In last decades, some analytical approximate methods and their modifications were suggested by researchers for solving nonlinear problems such as Adomian decomposition method [6, 7, 8], variational iteration method (VIM)

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[8, 9, 10,13], differential transform method [8, 11], homotopy perturbation method [12, 13, 14], harmonic balance method [15], incremental harmonic balance method [16], Newton-harmonic balance method [17], variational approach method [18], amplitude-frequency formulation [19-21], energy balance method [20], max-min approach [21]. Variety of methods to solve nonlinear problems provide more reliable results for the modal parameter on the vibration period. Approximate solutions of the rigid rod rocking back and forth on a circular surface without slipping were previously obtained by variational approach method, amplitude-frequency formulation, max-min approach, Hamiltonian approach, modified homotopy perturbation method, modified harmonic balance method, iteration perturbation method, parameter expansion method, energy balance method, residue harmonic balance method [22-39].

An analytical solution for the problem is not an easy task to obtain and sometimes it is not possible. The case is also the same for the analytical approximate solutions since they need analytical evaluation of the integrals that stem from highly nonlinear governing equation. While the analytical approximate solutions are expected to give more accurate results, the desired solution may not be obtained due to the integration of nonlinear functions that lead to enormous computation time. IVIM reduces the integration time drastically resulting from nonlinear terms in the governing equation and it is a great advantage over analytical approximation techniques. IVIM can be described as the numerical interpretation of the VIM. Although IVIM solution is obtained numerically, the method stems from an analytical approach. Hence, IVIM provides an analytical based numerical solution. The method was first proposed for the initial-value problems by Salkuyeh and Tavakoli [40]. Atay *et al.* [41] applied IVIM for the solution of stiff differential equations and Coşkun *et al.* [42] used the technique to solve jamming transition problem. This study investigates the application of IVIM for the first time to the motion of rigid rod rocking back and forth on a circular surface without slipping.

THEORY

The governing equation of motion of rigid rod rocking back and forth on a circular surface without slipping can be expressed as,

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{d^2u}{dt^2}+\frac{1}{16}u\left(\frac{du}{dt}\right)^2+\frac{g}{4l}u\cos u=0 \qquad (1)$$

with the initial conditions $u(0) = A, u'(0) = 0$ where A is amplitude, g is gravitational acceleration, l is length of the rod and t is time [1, 5, 22, 23, 31, 33, 35, 36, 39]. The problem is illustrated in Fig. 1 and the angle q in Fig.1. corresponds to the solution $u(t)$ in Eq.(1).

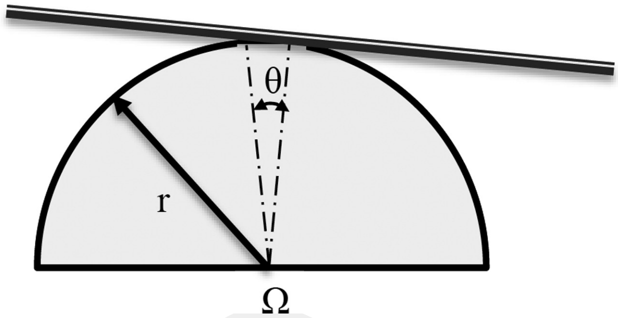

Figure 1. Geometry of rigid rod rocking back and forth on a circular surface.

The analytical approximation techniques that were used in the past to solve the problem [22-39] included the analytical integration process according to Eq.(1). However, the integration process with the nonlinear term u cos u do not provide a possible evaluation of a couple of successive integrations. Hence, cos u term is replaced with its Maclaurin series approximation with few terms. This transformation facilitates the analytical integration, but it's still very difficult to carry on computations through several successive orders of solution.

IVIM has the advantage of numerical integration that enables to integrate the nonlinear term u cos u directly instead of its series approximation. Numerical integration reduces the required time drastically and make it possible to calculate higher order solutions.

Below, the method and its application to the problem are explained and the results are discussed.

INTERPOLATED VARIATIONAL ITERATION METHOD (IVIM)

IVIM was first proposed by Salkuyeh and Tavakoli [40] to solve the following one-dimensional initial value problem. Below, the summary of the method is described based on the information given in their work [39].

$$u'(t)=f(t,u(t)),u(a)=u_a, t\in[a,T] \qquad (2)$$

VIM formulation [9] for the problem given in Eq. (2) may be written as follows:

$$u_{m+1}(t)=u_m(t)+\int_a^t \lambda(\xi,t)(u_m'(\xi)-f(\xi,u_m(\xi)))d\xi \qquad (3)$$

where / is Lagrangian multiplier. Application of integrating by parts to Eq. (3) $u_0(a) = 0$ assuming lead to following procedure.

$$u_{m+1}(t)=G_m(t)-\int_a^t H_m(\quad,t)d \qquad (4)$$

where

$$G_m(t) = (1+\lambda(t,t))u_m(t) - \lambda(a,t)(u_m(a) \tag{5}$$

$$\begin{aligned} H_m(\xi,t) &= \frac{d\lambda(\xi,t)}{d\xi}u_m(\xi) \\ &+ \lambda(\xi,t)f(\xi,u_m(\xi)) \end{aligned} \tag{6}$$

Dividing the time domain $[a,T]$ into n–1 subdomains discretizes the solution domain with the following nodal points.

$$t_i = a+(i-1)h,\ i=1,2,\dots,n \quad h=\frac{T-a}{n-1} \tag{7}$$

At this step, solution between nodes is interpolated by B-spline basis functions of first order and a numerical integration becomes possible with the piecewise linear interpolation to $H_m(\xi,t)$ in Eq. (4). The method results in a numerical equivalent of VIM formulation in Eq. (3).

$$\begin{aligned} u_{m+1}(t_i) \approx \hat{u}_{m+1}(t_i) &= G_m(t_i) - h\sum_{r=2}^{i-1} H_m(t_r,t_i) \\ &- \frac{h}{2}H_m(t_i,t_i) \end{aligned} \tag{8}$$

where index $m + 1$ denotes $m + 1^{st}$ order solution.

IVIM FOR THE MOTION OF RIGID ROD ON A CIRCULAR SURFACE

Governing equation (1) is replaced with a system of two first order equations.

$$\dot{u} - v = 0 \quad u(0) = A \tag{9}$$

$$\dot{v} + \frac{3u}{4+3u^2}v^2 + 4\frac{g}{L}\frac{3u}{4+3u^2}\cos u = 0 \qquad v(0)=0 \tag{10}$$

Lagrangian multiplier for Eqs. (9) can be determined by assuming $Lu = \dot{u}$ and imposing the variation with the restricted variation which simplifies to

$$\delta u_{m+1}(t) = \delta u_m(t) + \delta\int_0^t \lambda(\xi,t)\dot{u}_m(\xi)d\xi \tag{11}$$

$$\begin{aligned} \delta u_{m+1}(t) &= \delta u_m(t) + \lambda(\xi,t)\delta u_m(\xi)\Big|_{\xi=t} \\ &- \int_0^t \lambda'(\xi,t)\left(\int_0^\xi \delta\dot{u}_m(\tau)d\tau\right)d\xi \end{aligned} \tag{12}$$

Thus, the equations below are obtained.

$$\lambda'(\xi,t) = 0. \quad 1+\lambda(\xi,t)\Big|_{\xi=t} = 0 \tag{13}$$

Hence, Lagrangian multiplier for Eq.(9) becomes

$$\lambda_u(\xi,t) = -1 \tag{14}$$

Similarly, assuming $Lv = \dot{v}$, Lagrangian multiplier can be determined similarly for Eq. (10) as,

$$\lambda_v(\xi,t) = -1 \tag{15}$$

Then, IVIM would result in the following:

$$G_m^v(t) = 0 \tag{16}$$

$$\begin{aligned} H_m^v(\xi,t) &= -\frac{3u_m(\xi)}{4+3u_m(\xi)^2}v_m(\xi)^2 \\ &- 4\frac{g}{L}\frac{3u_m(\xi)}{4+3u_m(\xi)^2}\cos u_m(\xi) \end{aligned} \tag{17}$$

$$G_m^u(t) = A \tag{18}$$

$$H_m^u(\xi,t) = -v_m(\xi) \tag{19}$$

$$\hat{v}_{m+1}(t_i) = G_m^v(t_i) - h\sum_{r=2}^{i-1} H_m^v(t_r,t_i) - \frac{h}{2}H_m^v(t_i,t_i) \tag{20}$$

$$\hat{u}_{m+1}(t_i) = G_m^u(t_i) - h\sum_{r=2}^{i-1} H_m^u(t_r,t_i) - \frac{h}{2}H_m^u(t_i,t_i) \tag{21}$$

NUMERICAL RESULTS

Time domain for the computation is discretized by dividing time domain using different subintervals 0.0100, 0.0050, 0.0025 and 0.0010 secs. Four different cases, i.e., g/L = 1,2,5,10 are considered with all subintervals for each case. For each case, relative error (RE) for the solution is calculated according to the equation given below.

$$\text{Relative Error (RE) (\%)} = \left|\frac{\textit{Approximate Solution} - \textit{Exact Solution}}{\textit{Exact Solution}}\right| \times 100 \tag{22}$$

Convergence of solution for the vibration period by reducing the time step is shown in between Tables 1 – 4. IVIM solution for the period is obtained by determining the time between two successive peaks while the exact value is taken from the equation below [36].

$$T_{ex}(A) = 4\left(\frac{L}{3g}\right)^{1/2} \int_0^{\pi 2} \left\{ \frac{\left(4+3A^2\sin^2 t\right)A^2\cos^2 t}{8\left[A\sin A + \cos A - A\sin t \sin\left(A\sin t\right) - \cos\left(A\sin t\right)\right]} \right\}^{1/2} dt \quad (23)$$

Vibration periods for $g/L = 1$ are also compared in Table 5 with previous results obtained via Newton harmonic balance method (NHBM), variational approach method (VAM) [22], energy balance method (EBM) [33], amplitude-frequency formulation (AFF) [24], and residue harmonic balance method (RHBM) [36].

Time variation of solution for $A = 0.3\,\pi$ is given between Figures 2 – 5 for $g/L = 1,2,5,10$ respectively.

As stated previously, vibration periods for different values of g / L with different amplitudes are tabulated between Tables 1 – 4. Even for the largest time step

Table 1. The results of IVIM, Exact and relative errors in different amplitudes for case g/L = 1

Amplitude A	Δt_1 (0.01) (RE%)	Δt_2 (0.005) (RE%)	Δt_3 (0.0025) (RE%)	Δt_4 (0.0010) (RE%)	Exact
0.05π	3.67000 (0.23784)	3.66500 (0.10128)	3.66250 (0.03300)	3.66200 (0.01934)	3.66129
0.1π	3.77000 (0.16018)	3.76500 (0.02734)	3.76500 (0.02734)	3.76400 (0.00077)	3.76397
0.15π	3.95000 (0.22977)	3.94500 (0.10289)	3.94250 (0.03946)	3.94100 (0.00139)	3.94095
0.2π	4.21000 (0.16838)	4.20500 (0.04941)	4.20500 (0.04941)	4.20300 (0.00183)	4.20292
0.25π	4.57000 (0.00888)	4.57000 (0.00888)	4.57000 (0.00888)	4.57000 (0.00888)	4.56959
0.3π	5.08000 (0.05353)	5.08000 (0.05353)	5.07750 (0.00429)	5.07800 (0.01414)	5.07728
0.35π	5.80000 (0.03951)	5.80000 (0.03951)	5.80000 (0.03951)	5.79800 (0.00501)	5.79771
0.4π	6.90000 (0.06010)	6.90000 (0.06010)	6.89750 (0.02385)	6.89600 (0.00209)	6.89586

Table 2. The results of IVIM, Exact and relative errors in different amplitudes for case g/L = 2

Amplitude A	Δt_1 (0.01) (RE%)	Δt_2 (0.005) (RE%)	Δt_3 (0.0025) (RE%)	Δt_4 (0.0010) (RE%)	Exact
0.1π	2.67000 (0.31826)	2.66500 (0.13040)	2.66250 (0.03647)	2.66200 (0.01768)	2.66153
0.2π	2.98000 (0.27203)	2.97500 (0.10379)	2.97250 (0.01967)	2.97200 (0.00284)	2.97192
0.3π	3.60000 (0.27351)	3.59500 (0.13424)	3.59250 (0.06460)	3.59100 (0.02282)	3.59018
0.4π	4.88000 (0.07985)	4.88000 (0.07985)	4.87750 (0.02858)	4.87700 (0.01833)	4.87611

Table 3. The results of IVIM, Exact and relative errors in different amplitudes for case g/L = 5

Amplitude A	Δt_1 (0.01) (RE%)	Δt_2 (0.005) (RE%)	Δt_3 (0.0025) (RE%)	Δt_4 (0.0010) (RE%)	Exact
0.1π	1.69000 (0.39809)	1.68500 (0.10105)	1.68500 (0.10105)	1.68400 (0.04164)	1.68330
0.2π	1.88000 (0.02105)	1.88000 (0.02105)	1.88000 (0.02105)	1.88000 (0.02105)	1.87960
0.3π	2.28000 (0.41268)	2.27500 (0.19248)	2.27250 (0.08238)	2.27100 (0.01631)	2.27063
0.4π	3.09000 (0.19714)	3.08500 (0.03501)	3.08500 (0.03501)	3.08400 (0.00258)	3.08392

Table 4. The results of IVIM, Exact and relative errors in different amplitudes for case g/L = 10

Amplitude A	Δt_1 (0.01) (RE%)	Δt_2 (0.005) (RE%)	Δt_3 (0.0025) (RE%)	Δt_4 (0.0010) (RE%)	Exact
0.1π	1.20000 (0.81728)	1.19500 (0.39721)	1.19250 (0.18717)	1.19100 (0.06115)	1.19027
0.2π	1.33000 (0.06914)	1.33000 (0.06914)	1.33000 (0.06914)	1.33000 (0.06914)	1.32908
0.3π	1.61000 (0.27544)	1.61000 (0.27544)	1.60750 (0.11973)	1.60600 (0.02631)	1.60558
0.4π	2.19000 (0.42827)	2.18500 (0.19898)	2.18250 (0.08433)	2.18100 (0.01555)	2.18066

Table 5. Comparison between the IVIM and other methods for case g/L = 1

A	NHBM (RE%)	VAM (RE%)	EBM (RE%)	AFF (RE%)	RHBM (RE%)	IVIM (Δt_4) (RE%)	Exact
0.05π	3.66129 (.0054)	3.66129 (.0054)	3.66129 (.0054)	3.66129 (.0054)	3.66129 (.0054)	3.66200 (.0193)	3.66129
0.1π	3.76395 (.0005)	3.76394 (.0008)	3.76397 (.0008)	3.76394 (.0008)	3.76397 (.0000)	3.76400 (.0008)	3.76397
0.15π	3.94065 (.0053)	3.94064 (.0056)	3.94064 (.0056)	3.94062 (.0061)	3.94095 (.0023)	3.94100 (.0014)	3.94095
0.2π	4.20117 (.0416)	4.20116 (.0419)	4.20181 (.0264)	4.20105 (.0445)	4.20297 (.0012)	4.20300 (.0018)	4.20292
0.25π	4.56247 (.1534)	4.56246 (.1536)	4.56432 (.1129)	4.56194 (.1650)	4.56988 (.0088)	4.57000 (.0089)	4.56959
0.3π	5.05358 (.4668)	5.05355 (.4674)	5.05831 (.3736)	5.05162 (.5054)	5.07843 (.0226)	5.07800 (.0141)	5.07728
0.35π	5.72597 (1.2372)	5.72584 (1.0399)	5.73741 (1.0399)	5.71939 (1.3507)	5.80146 (.0648)	5.79800 (.0050)	5.79771
0.4π	6.67838 (3.1538)	6.67785 (3.1615)	6.70586 (2.7553)	6.67785 (3.1615)	6.90674 (0.1578)	6.89600 (.0021)	6.89586

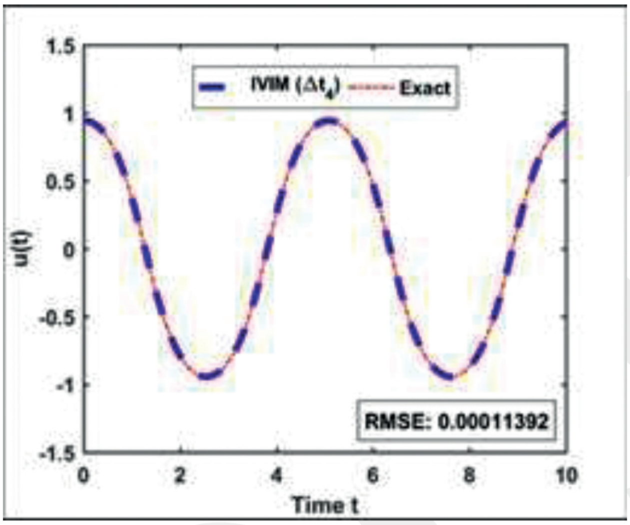

Figure 2. Variation of u(t) for g/L = 1 and A = 0.3π.

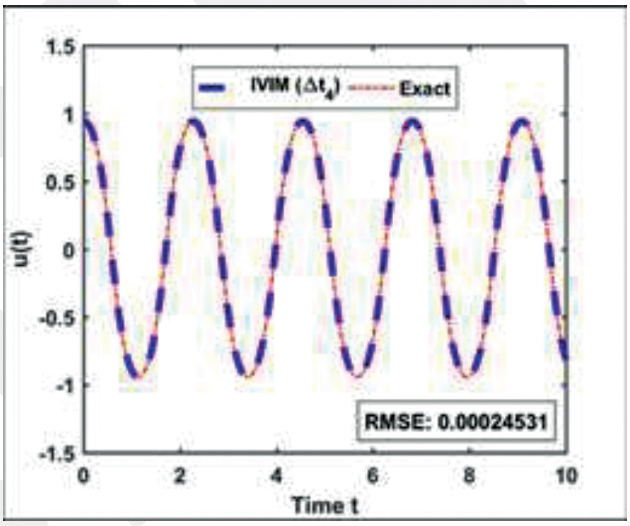

Figure 4. Variation of u(t) for g/L = 5 and A = 0.3π.

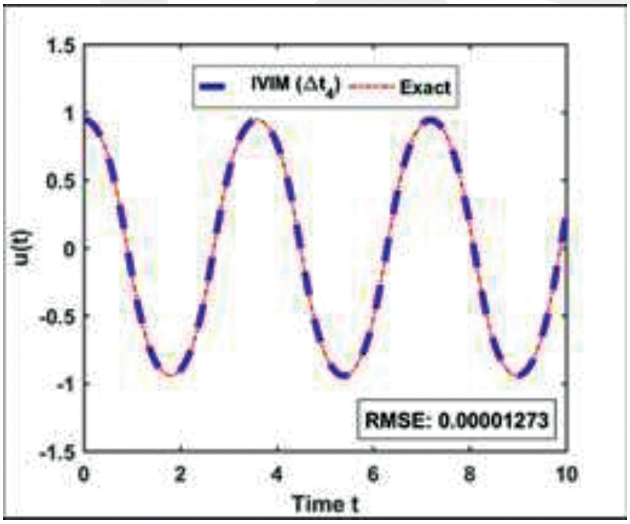

Figure 3. Variation of u(t) for g/L = 2 and A = 0.3π.

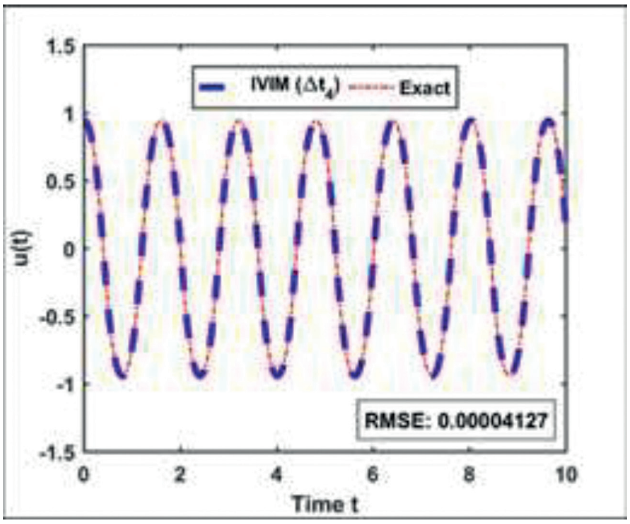

Figure 5. Variation of u(t) for g/L=10 and A = 0.3π.

relative error is less than one percent for all the cases g/L = 1,2,5,10 and for all amplitudes of each case. Reducing time steps provides a rapid convergence. The smallest time step used in the study is 0.0010 secs and the lowest relative error is obtained as 0.0000209 in the case $g/L = 1$ with amplitude of 0.4π. For the same step size the peak relative error is 0.0006919 in the case $g/L = 10$ with the amplitude 0.2π. A relative error of the same order is also occurred for the same case with the amplitude 0.1π. These results are reasonable and computed periods for the smallest step size are almost equal to analytical values. Order of solution was raised to twenty to fifty in the analysis and numerical experiments did not result in a significant change after twentieth order. Such orders of solution are impossible to maintain with VIM or other analytical approximate solution technique which is a great advantage of IVIM compared to analytical techniques.

Vibration periods calculated using IVIM are also compared in Table 5 with the available solutions existing in the literature. While the amplitude is increasing, the relative error due to NHBM, VAM, EBM, AFF, RHBM increases since the solutions via these methods were of low order because of high nonlinearity which made the analytical integration impossible after a few successive approximations. However, IVIM enables researchers to conduct the solution up to any order and numerical integration decreases the solution time in order of seconds. In addition, the results obtained from IVIM are very close to analytical solutions due to the analytical based formulation of the method.

Time variation of solutions and root mean square errors (RMSE) for all the cases g/L = 1,2,5,10 with the amplitude 0.3π are depicted between Figs.2 – 5 and the results are in excellent agreement with the analytical solutions.

CONCLUSION

In this study, the motion of rigid rod rocking back and forth on a circular surface without slipping is considered. The problem is analyzed using IVIM which may be described as the discretized form of analytical variational iteration technique. The solutions computed using the presented method compared with the analytical approximate results of the previous studies. The numerical evaluation of integrals significantly reduces the required time for the solution and the solution converges rapidly by a fine discretization in the solution domain. Numerical results show how the solution method is efficient and accurate which suggest the use of the approach for other nonlinear oscillation problems compared to analytical approximation techniques applicable to such problems that include lots of analytical operations.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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