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Twist-Bend Instability of a Cantilever Beam Subjected to an End Load via Homotopy Perturbation Method

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Abstract. In this article, twist-bend buckling analysis of a cantilever beam subjected to a concentrated end load is conducted using Homotopy Perturbation Method (HPM). Even in the linear stability analysis, obtaining an exact solution for some cases is not an easy task. However, by the use of HPM this difficulty can be overcome easily. This issue is presented with a case study and the results show that HPM can be used successfully in the analysis of twist-bend buckling of beams.

INTRODUCTION

In order to present the efficiency of the proposed technique, a case study is chosen for which an exact solution is available. To this aim, a cantilever beam subjected to a concentrated end load is considered. Den Hartog (1952) provided a closed-form solution for the problem. The load is assumed to be acting on the centerline to beam's stiff plane and the height of the beam is assumed as considerably larger than its thickness. The origin is located to the application point of the force. Let u be the sidewise displacement and φ be the angle of rotation (See Fig. 1).

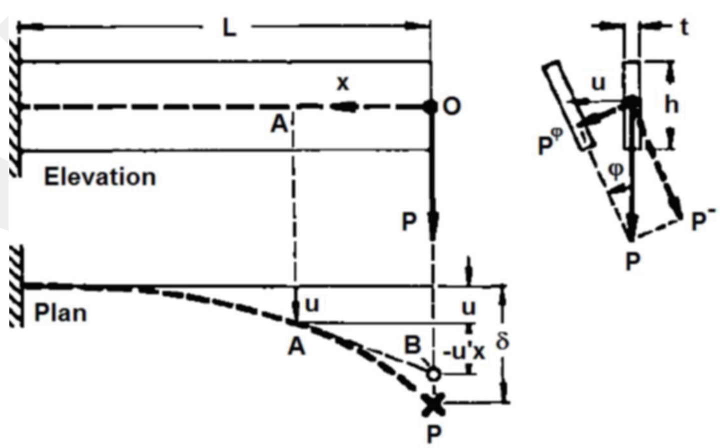

FIGURE 1. Cantilever beam with a concentrated end load (After Den Hartog, 1952)

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The end force P is resolved into components along and across the section at point x. There is a twisting couple at A of magnitude $P(\delta + u'x - u)$.

The differential equations of bending and twist at a small section dz at point A are as follows:

$$P(\delta + u'x - u) = -C\varphi' \tag{1}$$

$$P\varphi x = +EI_f u'' \tag{2}$$

where C is the torsional stiffness $GHt^3/3$ and EI_f is flexural stiffness $Eht^3/12$. The signs are due to the direction of x which is positive toward the left. Differentiating Eq.(1) produces the following.

$$P(u''x + u' - u') = -C\varphi'' \tag{3}$$

Then u is eliminated from Eq.(1) as

$$u'' = -C\varphi''/Px \tag{4}$$

One can obtain another equation for u'' as below.

$$u'' = P\varphi x/EI_f \tag{5}$$

Equating Eqs.(4) and (5) we find

$$\varphi'' + k^2x^2\varphi = 0 \quad k^2 = P^2/C.EI_f \tag{6}$$

Den Hartog (1952) described Eq.(6) as a nonlinear one with no simple solution and provided the following solution.

$$\varphi(x) = \sqrt{x}\left[c_1 J_{1/4}\left(\frac{kx^2}{2}\right) + c_2 Y_{1/4}\left(\frac{kx^2}{2}\right)\right] \tag{7}$$

The boundary conditions (BCs) associated with the problems are

$$\varphi'|_{x=0} = 0 \quad \varphi|_{x=L} = 0 \tag{8}$$

In order to apply BCs to Eq.(7), derivative of φ should be produced

$$\begin{aligned}\varphi'(x) = c_1 &\left[\frac{1}{2\sqrt{x}} J_{1/4}\left(\frac{kx^2}{2}\right) + \frac{1}{2}kx^{3/2}\left(J_{-3/4}\left(\frac{kx^2}{2}\right) - J_{5/4}\left(\frac{kx^2}{2}\right)\right)\right] \\ &+ c_2\left[\frac{1}{2\sqrt{x}} Y_{1/4}\left(\frac{kx^2}{2}\right) + \frac{1}{2}kx^{3/2}\left(Y_{-3/4}\left(\frac{kx^2}{2}\right) - Y_{5/4}\left(\frac{kx^2}{2}\right)\right)\right]\end{aligned} \tag{9}$$

It is very difficult to extract a characteristic value from Eqs.(7) and (9) using BCs provided in Eq.(8). Den Hartog (1952) obtained a solution in terms of power series given in the following.

$$\varphi(x) = a_0\left(1 - \frac{k^2x^4}{12} + \frac{k^4x^8}{672} - \frac{k^6x^{12}}{88704} + \cdots\right) + a_1 x\left(1 - \frac{k^2x^4}{20} + \frac{k^4x^8}{1440} - \frac{k^6x^{12}}{224640} + \cdots\right) \tag{10}$$

After applying the BCs to Eq.(10), Den Hartog (1952) obtained a three-term approximation as

$$kL^2 = 4.17 \text{ and } kL^2 = 6.2 \tag{11}$$

The first value is for the first buckling mode and the second value is for second buckling mode. Including a number of additional terms improves the first value to 4.01 (Den Hartog, 1952). Hence, Eq.(10) provides a critical value for P as follows:

$$P_{critical} = 4.01\frac{\sqrt{C.EI_f}}{L^2} \tag{12}$$

In the following part, application of HPM is presented to the problem at hand.

BASIC IDEAS OF HPM

To illustrate the basic idea, a general nonlinear differential equation is considered

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega \tag{13}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \tag{14}$$

where B is a boundary operator, $f(r)$ is a known analytic function, L is a linear operator and N is a nonlinear operator. By using HPM one can construct a homotopy $v(r,p)\colon \Omega \times (0,1) \rightarrow R$ that satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p\,[L(v) + N(v) - f(r)] = 0\,, \qquad p \in (0,1) \tag{15}$$

where $p \in (0,1)$ is an embedding parameter and u0 is an initial approximation which satisfies boundary conditions. While p changes from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$. The method provides a solution in terms of a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{16}$$

and setting p equals unity results in following approximate solution.

$$u = \lim_{p\rightarrow 1} v = v_0 + v_1 + v_2 + \cdots \tag{17}$$

The reader can refer to He (2003, 2006) who developed the technique for additional information.

APPLICATION OF HPM TO THE PROBLEM

Coskun (2010) applied HPM to tilt-buckling of Euler columns with variable stiffness. Same approach will be used for the governing equation of the problem given in Eq.(6).

$$\varphi'' + k^2 x^2 \varphi = 0 \;\; k^2 = P^2/C.EI_f$$

By taking d^2/dx^2 as linear operator and x^2 as nonlinear operator, HPM produces following successive approximation to the problem.

$$v_0'' - u_0'' = 0 \tag{18}$$

$$v_1'' + u_0'' + k^2 v_0 = 0 \; (19)$$

$$v_n'' + k^2 v_{n-1} = 0\,, \qquad n \geq 2 \; (20)$$

Inıtial approximation is chosen as the solution for linear operator L which is a first order polynomial and successive approximations are conducted up to twentieth iteration.

NUMERICAL RESULTS

Application of HPM with five, ten, fifteen and twenty term produces the following results for buckling modes.

TABLE 1. Convergence of kL^2 through the application of HPM only.

Order of HPM Solution	Mode 1	Mode 2	Mode 3
5	4.0126	-	-
10	4.0126	10.2461	16.7748
15	4.0126	10.2461	16.5159
20	4.0126	10.2461	16.5159

Den Hartog (1952) obtained the value as *4.01* with *5* term in the power series solution given in Eq.(10). As one can see from Eqs (18-20) HPM formulation can be implemented easily and Table 1 show that the proposed technique quickly converges to a solution. In order to provide a comparison, series solution is repeated with different number of terms and a *21-Term* solution produces the following results as exact solution.

TABLE 2. Power series solution for kL^2 with *21-term.*

# of terms	Mode 1	Mode 2	Mode 3
21	4.0126	10.2461	16.5159

As one can see form Tables 1 and 2, HPM produces excellent results for the problem presented in this study. HPM overcomes the difficulties for obtaining accurate results. Hence, the method is a good alternative to obtain a semi-analytical result for the stability problems of beams under twist-bend buckling.

CONCLUSION

In this study, twist-bend buckling of a cantilever beam with an end load at its end is considered. Beam is assumed as rectangular and very stiff in the direction of applied load, i.e., its height is considerably larger than its width. Although an analytical solution does exist, it is very difficult to solve characteristic value problem with this solution. However, HPM provides excellent results and it is an easy-to-use and efficient technique in the solution of such problems. Hence proposed method can be used effectively in the stability analysis of beams under twist-bend buckling.

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