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Eigenvalue problems for nonlinear third-order *m*-point *p*-Laplacian dynamic equations on time scales

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This work deals with the existence and uniqueness of a nontrivial solution for the third-order *p*-Laplacian *m*-point eigenvalue problems on time scales. We find several sufficient conditions of the existence and uniqueness of nontrivial solution of eigenvalue problems when λ is in some interval. The proofs are based on the nonlinear alternative of Leray–Schauder. To illustrate the results, some examples are included. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: time scales; eigenvalue problem; *p*-Laplacian; nontrivial solutions; nonlinear alternative of Leray–Schauder; fixed point theorem

1. Introduction

The theory of dynamical systems on time scales is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and study of finite difference equations in the discrete case; see [1–13] and the references therein.

Anderson [14] studied the third-order nonlinear boundary value problem

$$x''' = f(t, x(t)), \qquad t_1 \leq t \leq t_3,$$
$$x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0.$$

He proved the existence of solutions to nonlinear problem by using the Krasnoselskii and Leggett–Williams fixed-point theorems.

Jiang and Agarwal [15] considered the following singular third-order boundary value problem

$$y''' = f(y), \qquad 0 < x < +\infty,$$
$$y(0) = 0, \qquad y(+\infty) = 1, \qquad y'(+\infty) = y''(+\infty) = 0,$$

where $f(y) = (1-y)^\lambda g(y)$, $\lambda > 0$, $g(y)$ is positive and continuous on $(0, 1]$. They had a unique solution by using *a priori* estimates.

Li [16] studied the existence of single and multiple positive solutions to the nonlinear singular third-order two-point boundary value problem

$$u'''(t) + \lambda a(t) f(u(t)) = 0, \qquad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = 0,$$

where λ is a positive parameter. Under various assumptions on a and f, he established intervals of the parameter λ, which yield the existence of at least one, at least two, and infinitely many positive solutions of the boundary value problem by using Krasnoselskii's fixed point theorem of cone expansion-compression type.

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Liu *et al.*[17] considered the existence of at least one or two nondecreasing positive solutions for the following singular nonlinear third-order differential equation

$$x'''(t)+\lambda\alpha(t)f(t,x(t))=0,\qquad a<t<b,$$
$$x(a)=x''(a)=x'(b)=0.$$

They used Green's function and the fixed-point theorem of cone expansion and compression type.

Sun [18] investigated the existence of positive solutions for the nonlinear singular third-order three-point boundary value problem

$$u'''(t)-\lambda a(t)F(t,(u(t))=0,\qquad 0<t<1,$$
$$u(0)=u'(\eta)=u''(1)=0.$$

He established various results on the existence of single and multiple positive solutions to boundary value problem by using a fixed point theorem of cone expansion-compression type due to Krasnoselskii.

Yao [19] considered the existence of a positive solution for a semipositone second-order boundary value problem

$$u''(t)=\lambda q(t)f(t,u(t),u'(t)),\qquad 0<t<1,$$
$$\alpha u(0)-\beta u'(0)=d,\quad u(1)=0,$$

where $d>0,\ \alpha\geq 0,\ \beta\geq 0,\ \alpha+\beta>0$, and $q(t)f(t,u,v)\geq 0$ on a suitable subset of $[0,1]\times[0,+\infty)\times(-\infty,+\infty)$. The proofs are based on the Leray–Schauder fixed point theorem and the localization method.

Yao [20] found a successively iterative scheme of positive solution for the nonlinear third-order two-point boundary value problem

$$u'''(t)+q(u''(t))f(t,(u(t))=0,\qquad \text{a.e. } t\in[0,1],$$
$$u(0)=A,\qquad u(1)=B,\qquad u''(0)=C.$$

The main tool is monotone iterative technique on Banach space.

Zhou and Ma [21] found the existence of positive solutions and established a corresponding iterative scheme for the following third-order generalized right-focal boundary value problem with p-Laplacian operator:

$$(\phi_p(u''))'(t)+q(t)f(t,(u(t))=0,\qquad 0\leq t\leq 1,$$
$$u(0)=\sum_{i=1}^{m}\alpha_i u(\xi_i),\qquad u'(\eta)=0,\qquad u''(1)=\sum_{i=1}^{n}\beta_i u''(\theta_i).$$

The main tool is the monotone iterative technique.

The boundary value problems with p-Laplacian operator [21–24] and higher-order nonlinear boundary value problems [14–19, 21] have been studied extensively in the literature. But, there are not much concerning third-order p-Laplacian dynamic equations on time scales, see [25].

In this paper, we consider the existence and uniqueness of a nontrivial solution for the third-order p-Laplacian m-point eigenvalue problems on time scales

$$(\phi_p(u^{\Delta\nabla}))^{\nabla}+\lambda f(t,u(t),u^{\Delta}(t))=0,\qquad t\in(0,T)_{\mathbb{T}},\tag{1.1}$$

$$u(0)=0,\quad u^{\Delta}(T)=\sum_{i=1}^{m-2}a_i u^{\Delta}(\xi_i),\quad u^{\Delta\nabla}(0)=0,\quad m\geq 3,\tag{1.2}$$

where $\phi_p(u)$ is p-Laplacian operator, that is, $\phi_p(u)=|u|^{p-2}u$, for $p>1$, with $(\phi_p)^{-1}=\phi_q$ and $1/p+1/q=1, \lambda>0$ is a parameter. Some basic knowledge and definitions about time scales can be found in [5, 6].

Motivated by the results mentioned earlier, in this paper, we shall show existence and uniqueness of nontrivial solutions of (1.1) and (1.2). Our proofs are based on the nonlinear alternative of Leray–Schauder.

The conditions we used in the paper are different from those in [15–17, 19, 20, 26]. For the methods used in [14, 16–18], $u^n(t)$ is linear with respect to u, and thus, the corresponding Green's function $G(t,s)$ exists. However, with regard to problems (1.1) and (1.2), when $p\neq 2,\ \phi_p(u)$ is not linear with respect to u, and thus, the corresponding Green's function $G(t,s)$ does not exist. So, the methods used in [14, 16–18] are not applicable to problems (1.1) and (1.2). When the nonlinear term does not satisfy the usual conditions used in [11, 21–23, 27, 28], we concentrate on the case. The results in the paper improve those presented in [25]. The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting.

The following conditions will be used in this paper:

(H1) $0,T\in\mathbb{T},\ 0<\xi_1<\xi_2<\cdots<\xi_{m-2}<\rho(T),\ a_i\in[0,\infty)$ satisfy $0\leq\sum_{i=1}^{m-2}a_i<1\qquad i=1,2,\ldots m-2;$
(H2) $f\in C_{ld}([0,T]\times R\times R)$.

2. Preliminaries and lemmas

For convenience, we list the following well-known definitions, which can be found in [5,6].

Definition 2.1
A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} | \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} | \tau < r\} \in \mathbb{T},$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2
For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the nabla derivative of f at t, denoted by $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon |\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3
A function f is left-dense continuous (ld continuous), if f is continuous at each ld point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. The set of ld-continuous functions f will be denoted by $C_{ld}(\mathbb{T})$.

Definition 2.4
If $G^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a).$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

Lemma 2.1
If $1 - \sum_{i=1}^{m-2} a_i \neq 0$, then for $h \in C_{ld}([0, T], R)$,

$$(\phi_p(u^{\Delta\nabla}))^{\nabla} + \lambda h(t) = 0, \qquad t \in (0, T), \tag{2.1}$$

$$u(0) = 0, \qquad u^{\Delta}(T) = \sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i), \qquad u^{\Delta\nabla}(0) = 0, \tag{2.2}$$

has the unique solution

$$\begin{aligned} u(t) = &- \int_0^t (t - s)\phi_q \left(\lambda \int_0^s h(\tau)\nabla\tau \right) \nabla s \\ &+ t \left[\frac{\int_0^T \phi_q \left(\lambda \int_0^s h(\tau)\nabla\tau \right) \nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q \left(\lambda \int_0^s h(\tau)\nabla\tau \right) \nabla s}{1 - \sum_{i=1}^{m-2} a_i} \right]. \end{aligned} \tag{2.3}$$

Proof
From (2.1), using the properties of the integral in Section 8.4 of Reference [5], we have

$$u(t) = - \int_0^t (t - s)\phi_q \left(\lambda \int_0^s h(\tau)\nabla\tau - C_1 \right) \nabla s + C_2 t + C_3. \tag{2.4}$$

Because $u^{\Delta\nabla}(0) = 0$ and $u(0) = 0$, we obtain $C_3 = 0,\ C_1 = 0$. We solve for C_2. By $u^{\Delta}(T) = \sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i)$, it follows that

$$-\int_0^T \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s + C_2 = \sum_{i=1}^{m-2} a_i \left[-\int_0^{\xi_i} \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s + C_2\right].$$

Solving the aforementioned equation, we have

$$C_2 = \frac{\int_0^T \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Substituting this in (2.4), we obtain

$$\begin{aligned} u(t) = &-\int_0^t (t-s)\phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s \\ &+ t\left[\frac{\int_0^T \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\lambda \int_0^s h(\tau)\nabla\tau\right)\nabla s}{1 - \sum_{i=1}^{m-2} a_i}\right]. \end{aligned}$$

It is easy to see that the BVP $(\phi_p(u^{\Delta\nabla}))^{\nabla} = 0,\ u(0) = 0,\ u^{\Delta}(T) = \sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i),\ u^{\Delta\nabla}(0) = 0$ has only the trivial solution. Hence, u in (2.3) is the unique solution of (2.1) and (2.2). This completes the proof. □

Let $X = C_{ld}[0, T]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0, T]$, and $\|u\| = \max_{t\in[0,T]} \mid u(t) \mid$ is defined as usual by the maximum norm. We present the norm $Y = C_{ld}^1[0, T]$ by

$$\|u\|_1 = \|u\| + \|u^{\Delta}\| = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |u^{\Delta}(t)|.$$

Obviously, it follows that $(Y, \|u\|_1)$ is a Banach space.

Lemma 2.2 ([29])
Let X be a real Banach space and Ω be a bounded open subset of $X, 0 \in \Omega,\ A : \overline{\Omega} \to X$ be a completely continuous operator. Then either there exist $x \in \partial\Omega,\ \lambda > 1$ such that $A(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.

3. Main results

For notational convenience, we denote

$$\begin{aligned} \varphi(s) &= \phi_q\left(\int_0^s (p(\tau) + q(\tau))\nabla\tau\right), \\ \psi(s) &= \phi_q\left(\int_0^s r(\tau)\nabla\tau\right), \end{aligned}$$

where p, q, r are nonnegative functions and $p, q, r \in L^1[0, T]$,

$$L_1 = \int_0^T (T-s)\varphi(s)\nabla s + \left(\frac{2 - \sum_{i=1}^{m-2} a_i + T}{1 - \sum_{i=1}^{m-2} a_i}\right)\int_0^T \varphi(s)\nabla s + \left(\frac{T+1}{1 - \sum_{i=1}^{m-2} a_i}\right)\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\nabla s,$$

$$L_2 = \int_0^T (T-s)\psi(s)\nabla s + \left(\frac{2 - \sum_{i=1}^{m-2} a_i + T}{1 - \sum_{i=1}^{m-2} a_i}\right)\int_0^T \psi(s)\nabla s + \left(\frac{T+1}{1 - \sum_{i=1}^{m-2} a_i}\right)\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \psi(s)\nabla s.$$

Theorem 3.1
Assume that (H1), (H2) hold, $f : [0, T] \times R \times R$ is ld-continuous $R = (-\infty, +\infty),\ f(t, 0, 0) \not\equiv 0$, for $t \in [0, T]$, there exist nonnegative functions $p, q, r \in L^1[0, T]$ such that

$$|f(t, u, v)| \le p(t)|u|^{p-1} + q(t)|v|^{p-1} + r(t), \quad \text{a.e. } (t, u, v) \in [0, T] \times R \times R, \tag{3.1}$$

and there exists $t_0 \in [0, T]$ such that $p(t_0) \ne 0$ or $q(t_0) \ne 0$. Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, problems (1.1) and (1.2) have at least one nontrivial solution $u^* \in C_{ld}^1([0, T], R)$.

Proof
In view of Lemma 2.1, we know that problems (1.1) and (1.2) have a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = Au(t) := -\int_0^t (t-s)\phi_q\left(\lambda\int_0^s f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\right)\nabla s + t\left[\frac{\int_0^T \phi_q\left(\lambda\int_0^s f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\right)\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\phi_q\left(\lambda\int_0^s f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i}\right],$$

in Y. Therefore, we only have to search for a fixed point of A in Y. It follows from the Arzela–Ascoli theorem on time scales [30] and Lebesgue's dominated convergence theorem on time scales [31] that $A : Y \to Y$ is completely continuous.

Because $|f(t,0,0)| \le r(t),\ a.e.\ t \in [0,T]$, we know that $\int_0^T \psi(s)\nabla s > 0$, from $p(t_0) \neq 0$ or $q(t_0) \neq 0$, we can easily find $\int_0^T \varphi(s)\nabla s > 0$. Let

$$n = \frac{L_2}{L_1},\ \Omega = \left\{u \in C^1_{ld}[0,T] : \|u\|_1 < n\right\}.$$

Assume $u \in \partial\Omega,\ \mu > 1$ such that $Au = \mu u$. Then

$$\mu n = \mu\|u\|_1 = \|Au\|_1 = \|Au\| + \|(Au)^{\Delta}\|.$$

Because

$$\begin{aligned}
\|Au\| = \max_{t\in[0,T]} |\, Au(t) \,| &\le \int_0^T (T-s)\phi_q\left(\lambda\int_0^s |\, f(\tau,u(\tau),u^{\Delta}(\tau)) \,|\,\nabla\tau\right)\nabla s \\
&\quad + T\frac{\int_0^T \phi_q\left(\lambda\int_0^s |f(\tau,u(\tau),u^{\Delta}(\tau))|\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\quad + T\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\lambda\int_0^s |f(\tau,u(\tau),u^{\Delta}(\tau))|\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\le \int_0^T (T-s)\phi_q\left(\lambda\int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\Delta}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s \\
&\quad + T\frac{\int_0^T \phi_q\left(\lambda\int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\Delta}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\quad + T\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\lambda\int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\Delta}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\le \int_0^T (T-s)\phi_q\left(\lambda\left[\|u\|_1^{p-1}\int_0^s (p(\tau)+q(\tau))\nabla\tau + \int_0^s r(\tau)\nabla\tau\right]\right)\nabla s \\
&\quad + T\frac{\int_0^T \phi_q\left(\lambda\left[\|u\|_1^{p-1}\int_0^s (p(\tau)+q(\tau))\nabla\tau + \int_0^s r(\tau)\nabla\tau\right]\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\quad + T\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q\left(\lambda\left[\|u\|_1^{p-1}\int_0^s (p(\tau)+q(\tau))\nabla\tau + \int_0^s r(\tau)\nabla\tau\right]\right)\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\le \int_0^T (T-s)\left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s \\
&\quad + T\frac{\int_0^T \left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&\quad + T\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s}{1-\sum_{i=1}^{m-2}a_i} \\
&= \phi_q(\lambda)\|u\|_1\left\{\int_0^T (T-s)\varphi(s)\nabla s + \frac{T\left[\int_0^T \varphi(s)\nabla s + \sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi(s)\nabla s\right]}{1-\sum_{i=1}^{m-2}a_i}\right\} \\
&\quad + \phi_q(\lambda)\left\{\int_0^T (T-s)\psi(s)\nabla s + \frac{T\left[\int_0^T \psi(s)\nabla s + \sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\psi(s)\nabla s\right]}{1-\sum_{i=1}^{m-2}a_i}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\|(Au)^{\triangle}\| = \max_{t\in[0,T]} |(Au)^{\triangle}(t)| &\le \int_0^T \phi_q\left(\lambda \int_0^s |f(\tau,u(\tau),u^{\triangle}(\tau))|\nabla\tau\right)\nabla s \\
&+ \frac{\int_0^T \phi_q\left(\lambda \int_0^s |f(\tau,u(\tau),u^{\triangle}(\tau))|\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\lambda \int_0^s |f(\tau,u(\tau),u^{\triangle}(\tau))|\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&\le \int_0^T \phi_q\left(\lambda \int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\triangle}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s \\
&+ \frac{\int_0^T \phi_q\left(\lambda \int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\triangle}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\left(\lambda \int_0^s \left[p(\tau)|u(\tau)|^{p-1} + q(\tau)|u^{\triangle}(\tau)|^{p-1} + r(\tau)\right]\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&\le \int_0^T \left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s \\
&+ \frac{\int_0^T \left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&+ \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \left[\phi_q(\lambda)\|u\|_1\varphi(s) + \phi_q(\lambda)\psi(s)\right]\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&= \phi_q(\lambda)\|u\|_1 \left\{\int_0^T \varphi(s)\nabla s + \frac{\int_0^T \varphi(s)\nabla s + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\nabla s}{1-\sum_{i=1}^{m-2} a_i}\right\} \\
&+ \phi_q(\lambda)\left\{\int_0^T \psi(s)\nabla s + \frac{\int_0^T \psi(s)\nabla s + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \psi(s)\nabla s}{1-\sum_{i=1}^{m-2} a_i}\right\},
\end{aligned}$$

we have

$$\|Au\|_1 \le \phi_q(\lambda)\|u\|_1 L_1 + \phi_q(\lambda) L_2.$$

Select $\lambda^* = \left(\frac{1}{2L_1}\right)^{p-1}$. Then when $0 < \lambda \le \lambda^*$, we obtain

$$\mu n = \mu\|u\|_1 = \|Au\|_1 \le \frac{1}{2L_1} L_1 \|u\|_1 + \frac{L_2}{2L_1}.$$

As a result,

$$\mu \le \frac{1}{2} + \frac{L_2}{2nL_1} = 1.$$

This contradicts $\mu > 0$, by Lemma 2.2, A has a fixed point $u^* \in \overline{\Omega}$, because $f(t,0,0) \not\equiv 0$, then when $0 < \lambda \le \lambda^*$, problems (1.1) and (1.2) have a nontrivial solution $u^* \in C^1_{ld}[0,T]$. The proof is complete. □

Corollary 3.1
Assume that (H1), (H2) hold, and $f : [0,T] \times R \times R \to (-\infty, 0]$ or $f : [0,T] \times R \times R \to [0,+\infty)$ is ld continuous, $f(t,0,0) \not\equiv 0$, for $t \in [0,T]$ and there exist nonnegative functions $p, q \in L^1[0,T]$ such that

$$\begin{aligned}
&|f(t,u_1,v_1) - f(t,u_2,v_2)| \le p(t)|u_1-u_2|^{p-1} + q(t)|v_1-v_2|^{p-1}, \\
&\text{a.e. } (t,u_i,v_i) \in [0,T] \times R \times R \ (i=1,2),
\end{aligned} \tag{3.2}$$

and there exists $t_0 \in [0,T]$ such that $p(t_0) \ne 0$ or $q(t_0) \ne 0$. Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, problems (1.1) and (1.2) have a unique nontrivial solution $u^* \in C^1_{ld}[0,T]$.

Proof
Indeed, let $u_2 = v_2 = 0$, then

$$|f(t,u_1,v_1)| \le p(t)|u_1|^{p-1} + q(t)|v_1|^{p-1} + |f(t,0,0)|, \quad \text{a.e. } (t,u_1,v_1) \in [0,T] \times R \times R.$$

We conclude from Theorem 3.1 that problems (1.1) and (1.2) have a nontrivial solution $u^* \in C^1_{ld}[0,T]$.

But in this case, we favor to concentrate on the uniqueness of a nontrivial solution for problems (1.1) and (1.2). Let A be given in Theorem 3.1, which will verify that this leads to a contradiction. Indeed,

$$
\begin{aligned}
\|Au_1 - Au_2\| &= \max_{t\in[0,T]} |Au_1(t) - Au_2(t)| \\
&= \max_{t\in[0,T]} \Bigg| -\int_0^t (t-s)\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\nabla s \\
&\quad + \int_0^t (t-s)\phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\nabla s \\
&\quad + \frac{t}{1-\sum_{i=1}^{m-2} a_i}\int_0^T \left[\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right. \\
&\quad \left. - \phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right]\nabla s \\
&\quad - \frac{t\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \left[\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right. \\
&\quad \left. - \phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right]\nabla s \Bigg| \\
&\le \int_0^T (T-s)\left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right. \\
&\quad \left. - \phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s \\
&\quad + \frac{T}{1-\sum_{i=1}^{m-2} a_i}\int_0^T \left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right. \\
&\quad \left. - \phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s \\
&\quad + \frac{T\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right. \\
&\quad \left. - \phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s \\
&\le \int_0^T (T-s)\phi_q(\lambda)\phi_q\left(\int_0^s \left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau)) - f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s \\
&\quad + \frac{T\int_0^T \phi_q(\lambda)\phi_q\left(\int_0^s \left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau)) - f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad + \frac{T\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \phi_q(\lambda) \\
&\quad \times \phi_q\left(\int_0^s \left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau)) - f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s \\
&\le \int_0^T (T-s)\phi_q(\lambda) \\
&\quad \times \phi_q\left(\int_0^s \left[p(\tau)|u_1(\tau)-u_2(\tau)|^{p-1} + q(\tau)\left|u_1^{\Delta}(\tau)-u_2^{\Delta}(\tau)\right|^{p-1}\right]\nabla\tau\right)\nabla s \\
&\quad + \frac{T}{1-\sum_{i=1}^{m-2} a_i}\int_0^T \phi_q(\lambda) \\
&\quad \times \phi_q\left(\int_0^s \left[p(\tau)|u_1(\tau)-u_2(\tau)|^{p-1} + q(\tau)\left|u_1^{\Delta}(\tau)-u_2^{\Delta}(\tau)\right|^{p-1}\right]\nabla\tau\right)\nabla s \\
&\quad + \frac{T\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \phi_q(\lambda) \\
&\quad \times \phi_q\left(\int_0^s \left[p(\tau)\mid u_1(\tau)-u_2(\tau)\mid^{p-1} + q(\tau)\mid u_1^{\Delta}(\tau)-u_2^{\Delta}(\tau)\mid^{p-1}\right]\nabla\tau\right)\nabla s \\
&\le \int_0^T (T-s)\phi_q(\lambda)\|u_1-u_2\|_1\varphi(s)\nabla s \\
&\quad + \frac{T}{1-\sum_{i=1}^{m-2} a_i}\int_0^T \phi_q(\lambda)\|u_1-u_2\|_1\varphi(s)\nabla s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{T\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}\int_0^{\xi_i}\phi_q(\lambda)|u_1-u_2\|_1\varphi(s)\nabla s\\
&=\phi_q(\lambda)\|u_1-u_2\|_1\left\{\int_0^T(T-s)\varphi(s)\nabla s\right.\\
&\left.+\frac{T}{1-\sum_{i=1}^{m-2}a_i}\left[\int_0^T\varphi(s)\nabla s+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi(s)\nabla s\right]\right\}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|(Au_1)^{\Delta}-(Au_2)^{\Delta}\| &= \max_{t\in[0,T]}|\,(Au_1)^{\Delta}(t)-(Au_2)^{\Delta}(t)\,|\\
&\le \int_0^T\left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right.\\
&\quad\left.-\phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\int_0^T\left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right.\\
&\quad\left.-\phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s\\
&\quad+\frac{\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}\int_0^{\xi_i}\left|\phi_q\left(\lambda\int_0^s f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\nabla\tau\right)\right.\\
&\quad\left.-\phi_q\left(\lambda\int_0^s f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\nabla\tau\right)\right|\nabla s\\
&\le \int_0^T\phi_q(\lambda)\phi_q\left(\int_0^s\left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\right.\right.\\
&\quad\left.\left.-f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\left[\int_0^T\phi_q(\lambda)\phi_q\left(\int_0^s\left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\right.\right.\right.\\
&\quad\left.\left.-f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s\\
&\quad+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_q(\lambda)\phi_q\left(\int_0^s\left|f(\tau,u_1(\tau),u_1^{\Delta}(\tau))\right.\right.\\
&\quad\left.\left.\left.-f(\tau,u_2(\tau),u_2^{\Delta}(\tau))\right|\nabla\tau\right)\nabla s\right]\\
&\le \phi_q(\lambda)\|u_1-u_2\|_1\left\{\int_0^T\varphi(s)\nabla s\right.\\
&\quad\left.+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\left[\int_0^T\varphi(s)\nabla s+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi(s)\nabla s\right]\right\}.
\end{aligned}
$$

Then

$$\|Au_1-Au_2\|_1\le\phi_q(\lambda)\|u_1-u_2\|_1L_1.$$

Selecting $\lambda^*=\left(\frac{1}{2L_1}\right)^{p-1}$, when $0<\lambda\le\lambda^*$, we obtain

$$\|Au_1-Au_2\|_1\le\frac{1}{2}\|u_1-u_2\|_1.$$

This is a contradiction. □

Theorem 3.2
Assume that (H1), (H2) hold, and $f:[0,T]\times R\times R\to R$ is ld continuous $R=(-\infty,+\infty)$, $f(t,0,0)\not\equiv 0$, for $t\in[0,T]$ and

$$0\le L=\limsup_{|u|+|v|\to+\infty}\max_{t\in[0,T]}\frac{|f(t,u,v)|}{|u|^{p-1}+|v|^{p-1}}<+\infty. \tag{3.3}$$

Then there exists a constant $\lambda^*>0$ such that for any $0<\lambda\le\lambda^*$, problems (1.1) and (1.2) have at least one nontrivial solution $u^*\in C^1_{ld}([0,T],R)$.

Proof
Let $0 < \epsilon < 1$ such that $L + 1 - \epsilon > 0$. By (3.3), there exists $H > 0$ such that

$$|f(t,u,v)| \le (L+1-\epsilon)(|u|^{p-1} + |v|^{p-1}), \quad |u| + |v| \ge H, \text{ for } t \in [0,T].$$

Let $K = \max_{t\in[0,T],|u|+|v|\le H} |f(t,u,v)|$. Then for any $(t,u,v) \in [0,T] \times R \times R$, we obtain

$$|f(t,u,v)| \le (L+1-\epsilon)(|u|^{p-1} + |v|^{p-1}) + K.$$

From Theorem 3.1, we know that problems (1.1) and (1.2) have at least one nontrivial solution $u^* \in C^1_{ld}([0,T],R)$.

Corollary 3.2
Assume that (H1), (H2) hold, and $f : [0,T] \times R \times R \to R$ defines that a function of three variables is ld continuous $R = (-\infty,+\infty)$, $f(t,0,0) \not\equiv 0$, for $t \in [0,T]$ and

$$0 \le L = \limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|u|^{p-1}} < +\infty,$$

or

$$0 \le L = \limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|v|^{p-1}} < +\infty.$$

Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, problems (1.1) and (1.2) have at least one nontrivial solution $u^* \in C^1_{ld}([0,T],R)$.

Remark 3.1
Theorem 3.2 and Corollary 3.2 include the case that f is jointly sublinear at $(-\infty,+\infty)$, that is,

$$\limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|u|^{p-1} + |v|^{p-1}} = 0,$$

or

$$\limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|u|^{p-1}} = 0,$$

or

$$\limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|v|^{p-1}} = 0.$$

□

4. Examples

In the section, we give some examples to explain our results. We only study the case $\mathbb{T} = R$, $(0,T) = (0,1)$.

Example 4.1
We consider the third-order eigenvalue problem

$$(\phi_3(u''))' + \lambda\left(\frac{u^2 t \sin t}{1+t^2} - t(\cos u')^2 + t(1+t)\right) = 0, \; t \in (0,1), \tag{4.1}$$

$$u(0) = 0, \quad u'(1) = \frac{1}{2}u'\left(\frac{3}{4}\right), \quad u''(0) = 0. \tag{4.2}$$

It is easy to check that $f : [0,T] \times R \times R \to R$ is continuous. In this case, $p = 3$, $m = 3$, $a_1 = \frac{1}{2}$, and $\xi_1 = \frac{3}{4}$. Noticing

$$\left|\frac{u^2 t \sin t}{1+t^2} - t(\cos u')^2 + t(1+t)\right| \le \frac{t}{t^2+1}|u|^2 + t|u'|^2 + t^2,$$

it follows from a direct calculation that

$$\varphi(s) = \phi_q\left(\int_0^s \left(\frac{\tau}{1+\tau^2} + \tau\right) d\tau\right) = \left[\frac{1}{2}\left(\ln(1+s^2) + s^2\right)\right]^{\frac{1}{2}},$$

$$
\begin{aligned}
L_1 &= \int_0^T (T-s)\varphi(s)ds + \left(\frac{2-\sum_{i=1}^{m-2} a_i + T}{1-\sum_{i=1}^{m-2} a_i}\right)\int_0^T \varphi(s)ds \\
&\quad + \left(\frac{T+1}{1-\sum_{i=1}^{m-2} a_i}\right)\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)ds \\
&= \int_0^1 (1-s)\left[\frac{1}{2}\left(\ln(1+s^2)+s^2\right)\right]^{\frac{1}{2}} ds \\
&\quad + \left(\frac{2-\frac{1}{2}+1}{1-\frac{1}{2}}\right)\int_0^1 \left[\frac{1}{2}\left(\ln(1+s^2)+s^2\right)\right]^{\frac{1}{2}} ds \\
&\quad + \left(\frac{1+1}{1-\frac{1}{2}}\right)\frac{1}{2}\int_0^{\frac{3}{4}} \left[\frac{1}{2}\left(\ln(1+s^2)+s^2\right)\right]^{\frac{1}{2}} ds \\
&\approx 0.1617 + 2.3858 + 0.5462 \approx 3.0937.
\end{aligned}
$$

Therefore, $\lambda^* = \left(\frac{1}{2L_1}\right)^2 \approx 0.0261$. Then by Theorem 3.1, we know that problems (4.1) and (4.2) have a nontrivial solution $u^* \in C^1([0,T],R)$ for any $\lambda \in (0,\lambda^*]$.

Example 4.2
We consider the third-order eigenvalue problem

$$(\phi_3(u''))' + \lambda(1+t) = 0,\ t \in (0,1), \tag{4.3}$$

$$u(0) = 0, \quad u'(1) = \frac{1}{2}u'\left(\frac{1}{2}\right), \quad u''(0) = 0. \tag{4.4}$$

It is easy to check that $f : [0,T]\times R\times R \to [0,\infty)$ is continuous. In this case, $p = 3$, $m = 3$, $a_1 = \frac{1}{2}$, and $\xi_1 = \frac{1}{2}$. Noticing

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| = |(1+t)-(1-t)| \le |u_1-u_2|^{3-1} + |v_1-v_2|^{3-1},$$

it follows from a direct calculation that

$$\varphi(s) = \phi_q\left(\int_0^s (1+1)d\tau\right) = (2s)^{\frac{1}{2}},$$

$$
\begin{aligned}
L_1 &= \int_0^T (T-s)\varphi(s)ds + \left(\frac{2-\sum_{i=1}^{m-2} a_i + T}{1-\sum_{i=1}^{m-2} a_i}\right)\int_0^T \varphi(s)ds \\
&\quad + \left(\frac{T+1}{1-\sum_{i=1}^{m-2} a_i}\right)\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)ds \\
&= \int_0^1 (1-s)(2s)^{\frac{1}{2}} ds + \left(\frac{2-\frac{1}{2}+1}{1-\frac{1}{2}}\right)\int_0^1 (2s)^{\frac{1}{2}} ds + \left(\frac{1+1}{1-\frac{1}{2}}\right)\frac{1}{2}\int_0^{\frac{1}{2}} (2s)^{\frac{1}{2}} ds \\
&\approx 0.3771 + 4.7141 + 0.6667 \approx 5.7579.
\end{aligned}
$$

Therefore, $\lambda^* = \left(\frac{1}{2L_1}\right)^2 \approx 0.0075$. Then by Corollary 3.1, we know that problems (4.3) and (4.4) have a unique nontrivial solution $u^* \in C^1([0,T],R)$ for any $\lambda \in (0,\lambda^*]$.

Example 4.3
We consider the third-order eigenvalue problem

$$(\phi_3(u''))' + \lambda\left(-u^{\frac{1}{2}} + t^2 \sin\sqrt{u^4+u'^2} + t^3(1-t)e^{\cos t}\right) = 0,\ t \in (0,1), \tag{4.5}$$

$$u(0) = 0, \quad u'(1) = \frac{1}{2}u'\left(\frac{1}{2}\right), \quad u''(0) = 0. \tag{4.6}$$

It is easy to check that $f : [0,T]\times R\times R \to (-\infty,+\infty)$ is continuous. In this case, $p = 3$, $m = 3$, $a_1 = \frac{1}{2}$, and $\xi_1 = \frac{1}{2}$. It is obvious that

$$\limsup_{|u|+|u'|\to+\infty} \max_{t\in[0,T]} \frac{\left|-u^{\frac{1}{2}} + t^2\sin\sqrt{u^4+u'^2} + t^3(1-t)e^{\cos t}\right|}{|u|^{3-1}+|u'|^{3-1}} = 0,$$

select $\epsilon = \frac{1}{2}$, it follows from a direct calculation that

$$\varphi(s) = \phi_q\left(\int_0^s \left(\frac{1}{2}+\frac{1}{2}\right)d\tau\right) = (s)^{\frac{1}{2}},$$

$$
\begin{aligned}
L_1 &= \int_0^T (T-s)\varphi(s)ds + \left(\frac{2-\sum_{i=1}^{m-2} a_i + T}{1-\sum_{i=1}^{m-2} a_i}\right)\int_0^T \varphi(s)ds \\
&\quad + \left(\frac{T+1}{1-\sum_{i=1}^{m-2} a_i}\right)\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)ds \\
&= \int_0^1 (1-s)(s)^{\frac{1}{2}}ds + \left(\frac{2-\frac{1}{2}+1}{1-\frac{1}{2}}\right)\int_0^1 (s)^{\frac{1}{2}}ds + \left(\frac{1+1}{1-\frac{1}{2}}\right)\frac{1}{2}\int_0^{\frac{1}{2}} (s)^{\frac{1}{2}}ds \\
&\approx 0.2667 + 3.3333 + 0.4714 \approx 4.0714.
\end{aligned}
$$

Therefore, $\lambda^* = \left(\frac{1}{2L_1}\right)^2 \approx 0.0151$. Then by Theorem 3.2, we know that problems (4.5) and (4.6) have a nontrivial solution $u^* \in C^1([0,T],R)$ for any $\lambda \in (0,\lambda^*]$.

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