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WSA-Supplements and Proper Classes

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Abstract: In this paper, we introduce the concept of wsa-supplements and investigate the objects of the class of short exact sequences determined by wsa-supplement submodules, where a submodule U of a module M is called a wsa-supplement in M if there is a submodule V of M with $U + V = M$ and $U \cap V$ is weakly semiartinian. We prove that a module M is weakly semiartinian if and only if every submodule of M is a wsa-supplement in M. We introduce CC-rings as a generalization of C-rings and show that a ring is a right CC-ring if and only if every singular right module has a crumbling submodule. The class of all short exact sequences determined by wsa-supplement submodules is shown to be a proper class which is both injectively and co-injectively generated. We investigate the homological objects of this proper class along with its relation to CC-rings.

Keywords: proper class of short exact sequences; wsa-supplement submodule; weakly semiartinian module; C-ring; CC-ring

MSC: 16D10; 18G25

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1. Introduction

Throughout this study, all rings considered are associative with an identity element and all modules at hand are right and unital. Given such a module M, we use the notations $E(M)$, $\mathrm{Soc}(M)$, $Z(M)$, $\mathrm{Rad}(M)$ for the injective hull, socle, singular submodule, and radical of M, respectively. The notation ($N \lneqq M$) $N \leq M$ means that N is a (proper) submodule of M. $Mod-R$ denotes the category of all right R-modules over a ring R. For the terminology and notations used in this work we refer the reader to [1–3].

For any $M \in Mod-R$, we denote the injectivity domain of M by $\mathfrak{In}^{-1}(M)$. It is clear that M is injective if and only if its injectivity domain is as large as it can be, that is, $\mathfrak{In}^{-1}(M) = Mod-R$. It is well known that every module is injective relative to any semisimple module. In [4], the authors introduced modules M whose injectivity domain $\mathfrak{In}^{-1}(M)$ is minimal possible, namely the class of all semisimple modules and called such modules *poor*. This definition gives a natural homological opposite to injectivity of modules since only injective modules have the class of all modules as their injectivity domain. It is proved in [5] (Proposition 1) that every ring has a poor module. However, semisimple poor modules need not exist over an arbitrary ring. Recall that a module M is said to *crumble* (or be a *crumbling* module) if $\mathrm{Soc}(M/N)$ is a direct summand of M/N for every submodule N of M. It follows from [5] (Corollary 2) that a module M crumbles if and only if it is a locally noetherian V-module. It is shown in [5] (Theorem 1) that a ring R has a semisimple poor module if and only if every right crumbling R-module is semisimple. Clearly, a ring R crumbles if and only if it is a right SSI-ring, that is, every semisimple right R-module is injective.

Following [6], we denote the sum of all submodules of a module M that crumble by $\mathrm{C}(M)$. By [6] (Propositions 3.1 and 3.4), $\mathrm{C}(M)$ is the largest submodule of M that crumbles and $\mathrm{Soc}(M) \leq \mathrm{C}(M)$. A module M is called *semiartinian* if $\mathrm{Soc}(M/N) \neq 0$ for every proper submodule N of M. As a proper generalization of artinian modules, the class of semiartinian modules are extensively studied in the literature. In [6], the authors considered modules of which factor modules have a nonzero crumbling submodule. A module M is called *weakly semiartinian* if $\mathrm{C}(M/N) \neq 0$ for every proper submodule N of M. The sum of all weakly semiartinian submodules of a module M is the largest weakly semiartinian submodule of M which we denote by $\mathrm{wsa}(M)$. Clearly, semiartinian modules and crumbling modules are examples of weakly semiartinian modules. A weakly semiartinian module need not be semiartinian, in general. An example of a weakly semiartinian module which is not semiartinian can be found in [6] (Remark 2). Various properties of weakly semiartinian modules are given in the same work.

It is well known that a module is semisimple if and only if its submodules are direct summands. As a generalization of direct summands, supplement submodules are defined as follows. Let M be a module and $U, V \leq M$. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently if $M = U + V$ and $U \cap V$ is small in V. Here a submodule S of a module M is called *small* in M, denoted by $S \ll M$, if $M \neq S + L$ for every proper submodule L of M. A module M is called *supplemented* if every one of its submodules has a supplement in M. Supplement submodules play an important role in ring theory and relative homological algebra. In recent years, types of supplement submodules are extensively studied by many authors. In a series of books and articles [1–3,7,8], the authors have obtained detailed information about variations of supplement submodules and related rings.

In [9], the author introduced proper classes to axiomatize conditions under which a class of short exact sequences of modules can be computed as Ext groups corresponding to a certain relative cohomology. The class $\mathcal{S}plit$ of all splitting short exact sequences of right R-modules and the class $\mathcal{A}bs$ of all short exact sequences of right R-modules are trivial examples of proper classes. It follows from [1] (20.7) that the class $\mathcal{S}upp$ of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi} N \longrightarrow K \longrightarrow 0$ such that $\operatorname{Im}\psi$ is a supplement in N is a proper class. Examples and properties of proper classes, especially related to supplements can be found in [10–12].

Recently defined type of supplement submodules is as follows. A submodule V of a module M is called an *sa-supplement* of U in M if $M = U + V$ and $U \cap V$ is semiartinian (see [7]). It is shown in [7] that the class $\mathcal{SAS}$ of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi} N \longrightarrow K \longrightarrow 0$ such that $\operatorname{Im}\psi$ is an sa-supplement in N is a proper class. Since semiartinian modules are weakly semiartinian, it is of interest to investigate a new type of supplement submodules by replacing the property of being "semiartinian" by being "weakly semiartinian". The purpose of this paper is to introduce the concept of wsa-supplement submodules and investigate the objects of the proper class determined by wsa-supplement submodules in relative homological algebra.

The paper is organized as follows. In Section 2, we prove that a module M is weakly semiartinian if and only if every submodule of M is a wsa-supplement in M. In particular, a ring R is weakly semiartinian if and only if every right maximal ideal of R is a wsa-supplement in R.

We introduce right CC-rings as a generalization of C-rings and give some characterizations of such rings in Section 3. We show that a ring R is a right CC-ring if and only if every singular right R-module has a crumbling submodule. A semilocal right CC-ring is a right C-ring. A right noetherian and a right WV-ring is a right CC-ring.

In Section 4, we show that, over an arbitrary ring, the class of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi} N \longrightarrow K \longrightarrow 0$ such that $\operatorname{Im}\psi$ is a wsa-supplement in N is a proper class. We study the objects of this class, which we call $\mathcal{WSS}$. We show that a module M is $\mathcal{WSS}$-co-injective if and only if it is a wsa-supplement $E(M)$. Over a right CC-ring, a projective module P is $\mathcal{WSS}$-co-injective if and only if $P/\mathrm{wsa}(P)$ is injective. A ring R is weakly semiartinian if and only if every right R-module is $\mathcal{WSS}$-co-injective.

Finally, we show that over a crumbling-free ring $\mathcal{WSS}$-coprojective modules are only the projective modules.

2. Weakly Semiartinian Modules

In this section, we give a characterization of weakly semiartinian modules via wsa-supplement submodules. Firstly, let us start by giving the closure properties.

Proposition 1 ([6] (Proposition 3.1)). *If $f : M \longrightarrow N$ is a homomorphism of modules, then $f(\mathrm{C}(M) \subseteq \mathrm{C}(N)$.*

Proposition 2. *The class of weakly semiartinian modules is closed under submodules, factor modules, direct sums, sums and extensions.*

Proof. By [6] (Propositions 3.1 and 3.4), we get that the class of weakly semiartinian modules is closed under submodules, factor modules, direct sums and sums. Let B be a module and A be a submodule of B with A and B/A weakly semiartinian. Assume that $\mathrm{C}(B/X) = 0$ for some $X \subsetneq B$. By Proposition 1, we have $\mathrm{C}(A/A \cap X) \cong \mathrm{C}((A + X)/X) \leq \mathrm{C}(B/X) = 0$. Since A is weakly semiartinian, $A/A \cap X) = 0$ so that $A \leq X$. $B/X \cong (B/A)/(X/A)$ is weakly semiartinian which implies that $\mathrm{C}(B/X) \neq 0$, a contradiction. Hence, B is weakly semiartinian. □

The sum of all weakly semiartinian submodules of a module M is denoted by $\mathrm{wsa}(M)$. By Proposition 2, $\mathrm{wsa}(M)$ is weakly semiartinian. Therefore M is weakly semiartinian if and only if $\mathrm{wsa}(M) = M$. Using this fact and Proposition 2, we have the following result.

Corollary 1. *For any module M,* $\mathrm{wsa}(M/\mathrm{wsa}(M)) = 0$.

Proof. Let $N \leq M$ containing $\mathrm{wsa}(M)$ such that $N/\mathrm{wsa}(M) \leq \mathrm{wsa}(M/\mathrm{wsa}(M))$. It follows from Proposition 2 that $N/\mathrm{wsa}(M)$ is weakly semiartinian. Since $\mathrm{wsa}(M)$ is weakly semiartinian, applying Proposition 2 once again, we obtain that N is weakly semiartinian. Therefore $N \subseteq \mathrm{wsa}(M)$. This means that $N/\mathrm{wsa}(M) = 0$. □

Let M be a module and $U \leq M$. We say that U is (has) a *weakly semiartinian supplement* (*wsa-supplement* for short) in M if there exists $V \leq M$ such that $U + V = M$ and $U \cap V$ is a weakly semiartinian module.

Theorem 1. *An R-module M is weakly semiartinian if and only if every submodule of M is a wsa-supplement in M.*

Proof. Necessity follows from Proposition 2. For sufficiency, suppose that $\mathrm{C}(mR) = 0$ for some $m \in M$. Let U be any submodule of mR. By the assumption, there exists a submodule V of M such that $M = U + V$ and $U \cap V$ is weakly semiartinian. Using modular law, we have $mR = U + V \cap mR$. Note that $\mathrm{C}(U \cap V) = \mathrm{C}(U \cap mR \cap V) \subseteq \mathrm{C}(mR) = 0$. It means that U is a direct summand of mR and so mR is semisimple. Therefore $mR = \mathrm{Soc}(mR) = \mathrm{C}(mR) = 0$, and hence $m = 0$. This completes the proof. □

A module M is said to be *crumbling-free* if $\mathrm{C}(M) = 0$. A ring R is called crumbling-free if R_R is crumbling free. Let R be a ring and A and B be R-modules. Recall that A is *B-injective* if for any submodule X of B, any homomorphism $f : X \to A$ extends to a homomorphism $g : B \to A$.

Proposition 3. *An R-module M is weakly semiartinian if and only if every crumbling-free R-module is M-injective.*

Proof. Necessity is clear since $\mathrm{C}(U) \neq 0$ for every submodule U of M. For sufficiency, suppose that N is a submodule of M with $\mathrm{C}(N) = 0$. Let $U \leq N$. Since N is crumbling-free, U is crumbling-free and so, by the hypothesis, U is M-injective. So we can write $M = U \oplus V$, where V is a submodule of M. By the modular law, we get $N = U \oplus N \cap V$. This means that $N = \mathrm{Soc}(N) = \mathrm{C}(N) = 0$. Hence M is weakly semiartinian. □

Proposition 4. *Let M be a module and U be a submodule of M with M/U weakly semiartinian. A submodule V of M is a wsa-supplement of U in M if and only if $M = U + V$ and V is weakly semiartinian.*

Proof. Let V be a wsa-supplement of U in M. Then $V/(U \cap V) \cong M/U$ is weakly semiartinian. Since $U \cap V$ is also weakly semiartinian, it follows from Proposition 2 that V is weakly semiartinian. The converse is clear by again Proposition 2. □

Since for a maximal submodule U of M we have M/U is simple, therefore weakly semiartinian, the following result is a consequence of Proposition 4.

Corollary 2. *Let M be a module and U be a maximal submodule of M. A submodule V of M is a wsa-supplement of U in M if and only if $M = U + V$ and V is weakly semiartinian.*

Recall that a module M is *coatomic* if every proper submodule of M is contained in a maximal submodule of M.

Corollary 3. *Let M be a coatomic module. Then M is weakly semiartinian if and only if every maximal submodule of M is a wsa-supplement in M.*

Proof. Necessity follows from Proposition 1. For sufficiency, assume that M is not weakly semiartinian, that is, $\mathrm{wsa}(M) \neq M$. Let N be a maximal submodule of M that contains $\mathrm{wsa}(M)$ and K be a wsa-supplement of N in M. Then K is weakly semiartinian by Corollary 2 and we have $K \leq \mathrm{wsa}(M) \leq N$ which implies $M = N + K \leq N$, contradicting the maximality of N. □

It is well known that a ring R is semisimple artinian if and only if every maximal right ideal of R is a direct summand of R. Now we give an analogous characterization of this fact for right weakly semiartinian rings.

Corollary 4. *A ring R is right weakly semiartinian if and only if every maximal right ideal of R is a wsa-supplement in R.*

3. A Generalization of C-Rings

In [1] (10.10), a ring R is called *a right C-ring* if for every right R-module M and for every proper essential submodule N of M, $\mathrm{Soc}(M/N) \neq 0$, that is M/N has a simple submodule. The class of right C-rings is studied by many authors in homological algebra. Semiartinian rings and Dedekind domains are examples right C-rings. Since semiartinian rings are weakly semiartinian, motivated by this fact, it is natural to introduce right CC-rings as follows: A ring R is called a right CC-ring if for every right R-module M and for every proper essential submodule N of M, $\mathrm{C}(M/N) \neq 0$, that is M/N has a cyclic crumbling submodule.

Proposition 5. *The following statements are equivalent for a ring R.*

1. *R is a right CC-ring;*
2. *Every singular right R-module has a cyclic crumbling submodule;*
3. *For every proper essential right ideal I of R, $\mathrm{C}(R/I) \neq 0$.*

Proof. $(1 \Rightarrow 2)$: Let M be a singular right R-module and $0 \neq m \in M$. Now consider the isomorphism $f : R/\mathrm{ann}(m) \longrightarrow mR$. Since M is singular, $\mathrm{ann}(m)$ is a non-zero proper essential right ideal of R. Then, $R/\mathrm{ann}(m)$ has a cyclic crumbling submodule, that is

$\mathrm{C}(R/\operatorname{ann}(m)) \neq 0$. It follows from Proposition 1 that $\mathrm{C}(mR) \neq 0$. This completes the proof of $(1 \Rightarrow 2)$.

$(2 \Rightarrow 3)$ is clear since R/I is a singular right R-module for every proper essential right ideal I of R.

$(3 \Rightarrow 1)$: Let M be an R-module and N be a proper essential submodule of M. We shall show that $\mathrm{C}(M/N) \neq 0$. Let $0 \neq m + N \in M/N$. Since M/N is singular, $\operatorname{ann}(m+N)$ is a proper essential right ideal of R. By assumption, $R/\operatorname{ann}(m+N)$ has a cyclic crumbling submodule. Applying Proposition 1, we obtain that $\mathrm{C}(R(m+N)) \neq 0$ and so $\mathrm{C}(M/N) \neq 0$. It means that R is a right CC-ring. □

As a consequence of Proposition 5, we have the following result.

Corollary 5. *Let R be commutative domain. Then the following statements are equivalent.*

1. *R is a right CC-ring;*
2. *Every torsion right R-module has a cyclic crumbling submodule.*

A ring R is called *a right weakly-V-ring* (WV-ring for short) if every simple right R-module is R/I-injective for any right ideal I of R such that R/I is proper. Clearly, every right V-ring is a right WV-ring. Since a right WV-ring need not be right noetherian; in general, the authors investigated when a right WV-ring is right noetherian in [13] and showed that a right WV-ring R is right noetherian if and only if every cyclic right R-module can be written as a direct sum of a projective module and a module which is either CS or right noetherian.

Proposition 6. *A right noetherian and a right WV-ring is a right CC-ring.*

Proof. Let R be a right noetherian and a right WV-ring. Suppose that N is a proper essential submodule of an R-module M. Let $0 \neq m + N \in M/N$. Then there exists a proper essential right ideal I of R such that $R/I \cong R(m+N)$. Clearly, $R(m+N)$ is noetherian. Since R is a right WV-ring, R/I is a V-module. It means that $R(m+N)$ crumbles and so M/N has a cyclic crumbling submodule. □

Proposition 7. *Let R be a ring with $R/\operatorname{Soc}(R_R)$ weakly semiartinian. Then R is a right CC-ring.*

Proof. By Proposition 5, it suffices to show that $\mathrm{C}(R/I) \neq 0$ for every proper essential right ideal I of R. Since $\operatorname{Soc}(R_R)$ is the intersection of all essential right ideals of R, $\operatorname{Soc}(R_R) \subseteq I$ and so $R/I \cong (R/\operatorname{Soc}(R_R))/(I/\operatorname{Soc}(R_R))$ is a weakly semiartinian R-module by Proposition 2. This means that $\mathrm{C}(R/I) \neq 0$. Hence R is a right CC-ring. □

A ring R is called *semilocal* if $R/\operatorname{Rad}(R)$ is semisimple. The class of semilocal rings properly contains the class of semiperfect rings. Note that over a semilocal ring a module with zero radical is semisimple (see [1]).

Proposition 8. *A semilocal and a right CC-ring is a right C-ring.*

Proof. Let I be a proper essential right ideal of R. Since R is a right CC-ring, we can write $\mathrm{C}(R/I) \neq 0$. Note also by [6] (Lemma 4) that $\operatorname{Rad}(\mathrm{C}(R/I)) = 0$. By [1] (17.2-3), we obtain that $\operatorname{Soc}(R/I) = \mathrm{C}(R/I) \neq 0$ since the ring is semilocal. This means that R is a right C-ring. □

Theorem 2. *Let R be a right CC-ring. Then an R-module M is semisimple if and only if $\operatorname{Soc}(M) = \operatorname{wsa}(M)$ and every essential submodule of M is a wsa-supplement in M.*

Proof. Necessity part is clear. For sufficiency, let U be a proper essential submodule of M. Then there is a wsa-supplement V of U in M, that is $U + V = M$ and $U \cap V$ is weakly semiartinian. Since R is a right CC-ring, $V/(U \cap V) \cong M/U$ is weakly semiartinian. Then V is weakly semiartinian by Proposition 2 and we have $V \leq \text{wsa}(M) = \text{Soc}(M) \leq U$. This implies $U = M$, a contradiction. Therefore, M has no proper essential submodules. Hence M is semisimple. □

4. The Objects of the Proper Class $\mathcal{WSS}$

In this section, we consider the class of short exact sequences determined by wsa-supplement submodules. Before doing so, here we give the definition of a proper class which plays a key role in relative homological algebra in terms of examining classes of short exact sequences along with their homological objects (see [9] for an equivalent definition of a proper class).

Definition 1. *Let $\mathcal{P}$ be a class of short exact sequences of right R-modules and R-module homomorphisms. If a short exact sequence $\mathbb{E} : 0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$ belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism *and g is said to be a $\mathcal{P}$-*epimorphism*.*

A subfunctor $\mathcal{P}$ of Ext is said to be a proper class if $\mathcal{P}(M, N)$ is a subgroup of $\text{Ext}(M, N)$ for every R-modules M, N, and one of the following conditions is satisfied.

1. The composition of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism whenever this composition is defined;
2. The composition of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism whenever this composition is defined.

Let R be a ring and $\mathcal{P}$ be a proper class of right R-modules. An R-module M is said to be *$\mathcal{P}$-injective* (resp., *$\mathcal{P}$-co-injective*) if $\text{Ext}_{\mathcal{P}}(K, M) = 0$ (resp., $\text{Ext}_{\mathcal{P}}(K, M) = \text{Ext}_R(K, M)$) for all right R-modules K. The smallest proper class for which every module from the class of modules $\mathcal{P}$ is co-injective is called *co-injectively generated* by $\mathcal{P}$.

A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$ is called WSS if $\text{Im}\, f$ is a wsa-supplement submodule of B. We denote the class of all WSS sequences by $\mathcal{WSS}$. The next result shows that the class $\mathcal{WSS}$ is a proper class over an arbitrary ring.

Proposition 9. *The class $\mathcal{WSS}$ is the proper class co-injectively generated by the class of weakly semiartinian modules.*

Proof. It follows from Proposition 2 and [14] (Theorem 2). □

Proposition 10. *The class $\mathcal{WSS}$ is injectively generated by the class of crumbling-free modules.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \mathcal{WSS}$, M be a crumbling-free module and $\alpha : A \longrightarrow M$ a homomorphism. Then $\alpha_* E : 0 \longrightarrow M \longrightarrow D \longrightarrow C \longrightarrow 0 \in \mathcal{WSS}$ since $\mathcal{WSS}$ is a proper class. Then there is a submodule K of D such that $M + K = D$ and $M \cap K$ is weakly semiartinian. By Proposition 1, we have $\text{C}(M \cap K) \leq \text{C}(M) = 0$ so that $\alpha_* E$ splits. Therefore, M is $\mathcal{WSS}$-injective.

Now let $F : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence such that every crumbling-free module is F-injective. Since $\text{C}(X/\text{wsa}(X)) = 0$, there is a submodule L of Y with $\text{wsa}(X) \leq L$ and $X/\text{wsa}(X) \oplus L/\text{wsa}(X) = Y/\text{wsa}(X)$. Then we have $X + L = Y$ and $X \cap L = \text{wsa}(X)$. Hence $F \in \mathcal{WSS}$. □

We call a module M *$\mathcal{WSS}$-co-injective*, if every short exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0,$$

of right R-modules starting with the module M is in the proper class $\mathcal{WSS}$. It follows that a module M is $\mathcal{WSS}$-co-injective if and only if it is a wsa-supplement in every extension.

It is clear that injective modules, semiartinian modules and wsa-supplementing modules are examples of $\mathcal{WSS}$-co-injective modules. Proposition 10 implies that a crumbling-free module is $\mathcal{WSS}$-co-injective if and only if it is injective. Recall that we denote the injective hull of a module M by $E(M)$.

Theorem 3. *The following statements are equivalent for a module M.*

1. *M is $\mathcal{WSS}$-co-injective;*
2. *M is a wsa-supplement in $E(M)$.*

Proof. $(1 \Rightarrow 2)$ is clear.

$(2 \Rightarrow 1)$: Let M be a wsa-supplement in $E(M)$ and let N be a module containing M. Since $E(M) \subseteq E(N)$, there exists a submodule $U \subseteq E(N)$ such that $E(N) = E(M) \oplus U$. Since M is a wsa-supplement in $E(M)$, M is a wsa-supplement in $E(N)$. Hence there exists a submodule V of $E(N)$ such that $E(N) = M + V$ and $M \cap V$ is weakly semiartinian. By modular law, we can write $N = N \cap E(N) = N \cap (M + V) = M + N \cap V$ and $M \cap (N \cap V) = (M \cap N) \cap V = M \cap V$ is weakly semiartinian. It means that M is $\mathcal{WSS}$-co-injective. □

The following result is a consequence of Theorem 3.

Corollary 6. *Let M be a module with $M/\operatorname{wsa}(M)$ injective. Then M is $\mathcal{WSS}$-co-injective.*

Proof. By the assumption, there exists a submodule K of $E(M)$ containing $\operatorname{wsa}(M)$ such that $M/\operatorname{wsa}(M) \oplus K/\operatorname{wsa}(M) = E(M)/\operatorname{wsa}(M)$. Therefore $M + K = E(M)$ and $M \cap K \subseteq \operatorname{wsa}(M)$. Applying Proposition 2, $M \cap K$ is weakly semiartinian and so M is a wsa-supplement in $E(M)$. It follows from Theorem 3 that M is $\mathcal{WSS}$-co-injective. □

The next result shows that the class of $\mathcal{WSS}$-co-injective modules is closed under extensions.

Proposition 11. *Let $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ be a short exact sequence of modules. If M and K are $\mathcal{WSS}$-co-injective, then so is N.*

Proof. By [15] (Proposition 1.9 and 1.14). □

Corollary 7. *Every finite direct sum of $\mathcal{WSS}$-co-injective modules is $\mathcal{WSS}$-co-injective.*

Proof. Let $n \in \mathbb{Z}^+$ and M_i $(1 \leq i \leq n)$ be any finite collection of $\mathcal{WSS}$-co-injective modules. Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. Suppose that $n = 2$, that is, $M = M_1 \oplus M_2$. Then $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ is a short exact sequence. Applying Proposition 11, we have that M is $\mathcal{WSS}$-co-injective. The proof is completed by induction on n. □

We do not know if any direct sum of $\mathcal{WSS}$-co-injective modules is $\mathcal{WSS}$-co-injective. Nevertheless, over right noetherian rings, we show that the class of $\mathcal{WSS}$-co-injective modules is closed under direct sums.

Theorem 4. *Let R be a right noetherian ring and $\{M_i\}_{i \in I}$ be a collection of $\mathcal{WSS}$-co-injective R-modules. Then $\bigoplus_{i \in I} M_i$ is $\mathcal{WSS}$-co-injective.*

Proof. Put $M = \bigoplus_{i \in I} M_i$. It is easy to see that $\operatorname{wsa}(M) = \bigoplus_{i \in I} \operatorname{wsa}(M_i)$. Since R is a right noetherian ring, $E(M)$ is the direct sum of $E(M_i)$ for each $i \in I$. Note that $E(M)/\operatorname{wsa}(M) = \bigoplus_{i \in I} E(M_i)/\bigoplus_{i \in I} \operatorname{wsa}(M_i) \cong \bigoplus_{i \in I}(E(M_i)/\operatorname{wsa}(M_i))$. Using Theorem 3, we can write $E(M_i)/\operatorname{wsa}(M_i) = (M_i/\operatorname{wsa}(M_i)) \oplus (K_i/\operatorname{wsa}(M_i))$ for some submodule $K_i/\operatorname{wsa}(M_i)$ of $E(M_i)/\operatorname{wsa}(M_i)$ $(i \in I)$. Let $K/\operatorname{wsa}(M) = \bigoplus_{i \in I} K_i/\operatorname{wsa}(M_i)$. Therefore $E(M)/\operatorname{wsa}(M) = M/\operatorname{wsa}(M) \oplus K/\operatorname{wsa}(M)$. This means that M is a wsa-supplement in $E(M)$. Applying Theorem 3 once again, we obtain that M is $\mathcal{WSS}$-co-injective. □

In general, a submodule of a $\mathcal{WSS}$-co-injective module need not be $\mathcal{WSS}$-co-injective. For example, the submodule $\mathbb{Z}_{\mathbb{Z}}$ of the $\mathcal{WSS}$-co-injective module $\mathbb{Q}_{\mathbb{Z}}$ is not $\mathcal{WSS}$-co-injective. We prove that every wsa-supplement submodule of a $\mathcal{WSS}$-co-injective module is $\mathcal{WSS}$-co-injective.

Proposition 12. *Let M be a $\mathcal{WSS}$-co-injective module and V be a wsa-supplement submodule of M. Then V is $\mathcal{WSS}$-co-injective.*

Proof. Let V be a wsa-supplement in M. Then $\mathbb{E} : 0 \longrightarrow V \longrightarrow M \longrightarrow M/V \longrightarrow 0$ is a short exact sequence in $\mathcal{WSS}$, that is, $U + V = M$ and $U \cap V$ is weakly semiartinian for some submodule U of M. Therefore by [15] (Proposition 1.8) V is $\mathcal{WSS}$-co-injective. □

The following fact is direct consequence of Proposition 12.

Corollary 8. *Every direct summand of a $\mathcal{WSS}$-co-injective module is $\mathcal{WSS}$-co-injective.*

We call a ring R weakly semiartinian if R_R is weakly semiartinian, or equivalently, if every R-module is weakly semiartinian.

Proposition 13. *The following statements are equivalent for a ring R.*

1. *R is right weakly semiartinian;*
2. *Every $\mathcal{WSS}$-co-injective R-module is weakly semiartinian;*
3. *Every injective R-module is weakly semiartinian.*

Proof. $(1 \Rightarrow 2)$ and $(2 \Rightarrow 3)$ are trivial.

$(3 \Rightarrow 1)$: R_R is a submodule of $E(R_R)$ which is weakly semiartinian by assumption. Proposition 2 completes the proof. □

A ring R is called *right hereditary* if every factor module of an injective module is injective. Now we prove that over right hereditary rings every factor module of a $\mathcal{WSS}$-co-injective module is $\mathcal{WSS}$-co-injective. Firstly, we need the following result.

Proposition 14. *$\mathcal{WSS}$-co-injective modules are closed under quotients if and only if quotients of injective modules are $\mathcal{WSS}$-co-injective.*

Proof. The necessity part follows from the fact that injective modules are $\mathcal{WSS}$-co-injective. For sufficiency, let M be a $\mathcal{WSS}$-co-injective module and N be a submodule of M. We have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & M/E(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M/N & \longrightarrow & E(M)/N & \longrightarrow & M/E(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array},$$

with exact rows and columns. Since M is $\mathcal{WSS}$-co-injective it has a wsa-supplement in $E(M)$. $\mathcal{WSS}$ being a proper class implies that M/N has a wsa-supplement in $E(M)/N$

which is $\mathcal{WSS}$-co-injective by assumption. By [15] (Proposition 1.8) M/N is $\mathcal{WSS}$-co-injective module. □

Corollary 9. *Let R be a right hereditary ring and M be a $\mathcal{WSS}$-co-injective R-module. Then every factor module of M is $\mathcal{WSS}$-co-injective.*

Proposition 15. *Let M be a $\mathcal{WSS}$-co-injective module. Then the following are equivalent:*

1. *$M/\operatorname{wsa}(M)$ is $\mathcal{WSS}$-co-injective;*
2. *$M/\operatorname{wsa}(M)$ is injective;*
3. *M/N is $\mathcal{WSS}$-co-injective for each weakly semiartinian submodule N of M;*
4. *M/N is $\mathcal{WSS}$-co-injective for each wsa-supplement submodule N of M.*

Proof. $(1 \Rightarrow 2)$ follows from Corollary 1.

$(2 \Rightarrow 3)$: Let N be a weakly semiartinian submodule of M. We have the short exact sequence $0 \longrightarrow \operatorname{wsa}(M)/N \longrightarrow M/N \longrightarrow M/\operatorname{wsa}(M) \longrightarrow 0$ with $M/\operatorname{wsa}(M)$ injective, hence $\mathcal{WSS}$-co-injective. By Proposition 2, weakly semiartinian modules are closed under quotients and so $\operatorname{wsa}(M)/N$ is $\mathcal{WSS}$-co-injective. By Proposition 11, M/N is also $\mathcal{WSS}$-co-injective.

$(3 \Rightarrow 4)$: Let N be a wsa-supplement submodule of M. Then there exists $K \leq M$ such that $N + K = M$ and $N \cap K$ is weakly semiartinian. Since $N \cap K \leq \operatorname{wsa}(M)$, we have the short exact sequence

$$0 \longrightarrow \operatorname{wsa}(M)/(N \cap K) \longrightarrow M/N \cap K \longrightarrow M/\operatorname{wsa}(M) \longrightarrow 0.$$

By Proposition 2, $\operatorname{wsa}(M)/(N \cap K)$ is $\mathcal{WSS}$-co-injective. $M/\operatorname{wsa}(M)$ is $\mathcal{WSS}$-co-injective by assumption. By Proposition 11, $M/(N \cap K)$ is also $\mathcal{WSS}$-co-injective. Since M/N is isomorphic to a direct summand of $M/(N \cap K)$, M/N is $\mathcal{WSS}$-co-injective module.

$(4 \Rightarrow 1)$ follows from the fact that $\operatorname{wsa}(M)$ is a wsa-supplement of M in M. By assumption $M/\operatorname{wsa}(M)$ is $\mathcal{WSS}$-co-injective. □

Corollary 10. *The following statements are equivalent:*

1. *$I/\operatorname{wsa}(I)$ is injective for every injective module I;*
2. *$M/\operatorname{wsa}(M)$ is injective for every $\mathcal{WSS}$-co-injective module M;*
3. *The class of $\mathcal{WSS}$-co-injective modules is closed under wsa-supplement quotients.*

Proof. The equivalence of 2 and 3 is given in Proposition 15 and $(2 \Rightarrow 1)$ is clear.

$(1 \Rightarrow 2)$: Let M be a $\mathcal{WSS}$-co-injective module. Then M has a wsa-supplement N in injective hull $E(M)$ of M. Since $M + N = E(M)$ and $M \cap N$ is weakly semiartinian, we have $M \cap N \leq \operatorname{wsa}(M)$ and hence $E(M)/\operatorname{wsa}(M) = [M/\operatorname{wsa}(M)] \oplus [(N + \operatorname{wsa}(M))/\operatorname{wsa}(M)]$. By Proposition 15, $E(M)/\operatorname{wsa}(M)$ is a $\mathcal{WSS}$-co-injective module and so is $M/\operatorname{wsa}(M)$ as a direct summand of $E(M)/\operatorname{wsa}(M)$. Corollary 8 completes the proof. □

Corollary 11. *Let R be a right CC-ring. Then the class of $\mathcal{WSS}$-co-injective modules is closed under wsa-supplement quotients.*

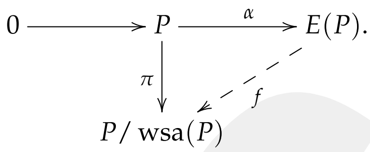
Proof. Let R be a right CC-ring and I be an injective module. Then every singular module is weakly semiartinian which implies that every crumbling-free module is nonsingular. Since $I/\operatorname{wsa}(I)$ is crumbling-free, it is nonsingular and it follows from [16] (Lemma 2.3) that $\operatorname{wsa}(I)$ is closed I. We have $I \cong \operatorname{wsa}(I) \oplus [I/\operatorname{wsa}(I)]$ and so $I/\operatorname{wsa}(I)$ is injective. The rest of the proof follows from Corollary 10. □

Proposition 16. *The following statements are equivalent for a projective module P.*

1. *P is $\mathcal{WSS}$-co-injective;*

2. *$P/\operatorname{wsa}(P)$ is a homomorphic image of an injective module;*
3. *There exists a weakly semiartinian submodule M of P such that P/M is a homomorphic image of an injective module.*

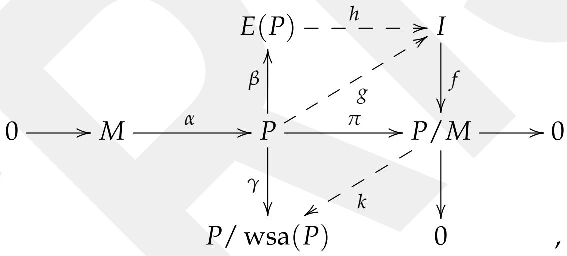
Proof. ($1 \Rightarrow 2$): Let $\alpha : P \to E(P)$ be the inclusion and $\pi : P \to P/\operatorname{wsa}(P)$ the canonical epimorphism. Then we have the diagram



Since P is $\mathcal{WSS}$-co-injective and $P/\operatorname{wsa}(P)$ is crumbling-free, it follows from Proposition 10 that there exists a homomorphism $f : E(P) \to P/\operatorname{wsa}(P)$ such that $f\alpha = \pi$. Since π is an epimorphism, then so is f. Hence $P/\operatorname{wsa}(P) = f(E(P))$.

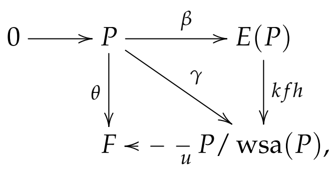
($2 \Rightarrow 3$): Since $\operatorname{wsa}(P)$ is weakly semiartinian, taking $M = \operatorname{wsa}(P)$ yields the result by assumption.

($3 \Rightarrow 1$): Let M be a weakly semiartinian submodule of P such that there is an epimorhism $f : I \to P/M$ with I injective. Consider the diagram



where $\alpha : M \to P$ and $\beta : P \to E(P)$ are inclusions and $\pi : P \to P/M$ and $\gamma : P \to P/\operatorname{wsa}(P)$ are canonical epimorphisms. Since M is weakly semiartinian, there is a homomorphism $k : P/M \to P/\operatorname{wsa}(P)$ such that $k\pi = \gamma$. Since f is an epimorphism and P is projective, there is a homomorphism $g : P \to I$ such that $fg = \pi$. Since β is a monomorphism and I is injective, there is a homomorphism $h : E(P) \to I$ such that $h\beta = g$. We have that the homomorphism $kfh : E(P) \to P/\operatorname{wsa}(P)$ satisfies $(kfh)\beta = k(f(h\beta)) = k(fg) = k\pi = \gamma$.

Now let F be a crumbling-free module and $\theta : P \to F$ be a homomorphism. Since $\operatorname{wsa}(P) \leq \operatorname{Ker}\theta$, by Factor Theorem there is homomorphism $u : P/\operatorname{wsa}(P) \to F$ such that $u\gamma = \theta$. Then, we have the diagram,



with the homomorphism $ukfh : E(P) \to F$ that satisfies $(ukfh)\beta = u((kfh)\beta) = u\gamma = \theta$ which implies by Proposition 10 that P is $\mathcal{WSS}$-co-injective. □

Corollary 12. *Every projective module is $\mathcal{WSS}$-co-injective if and only if every crumbling-free module is a homomorphic image of an injective module.*

Proof. For necessity let M be a crumbling-free module. There is an epimorphism $f : P \to M$ with P projective. Let $E(P)$ be the injective hull of P and $\alpha : P \to E(P)$ be the inclusion. Since P is $\mathcal{WSS}$-co-injective, it follows from Proposition 10 that there is a homomorphism

$g : E(P) \to M$ such that $g\alpha = f$. Clearly, f is an epimorphism. Sufficiency follows from Proposition 16. □

Corollary 13. *Over a right CC-ring, a projective module P is $\mathcal{WSS}$-co-injective if and only if $P/\operatorname{wsa}(P)$ is injective.*

Proof. For necessity, let P be a $\mathcal{WSS}$-co-injective module. Then, by Proposition 16, there is an epimorphism $f : I \to P$ for some injective module I. Since $P/\operatorname{wsa}(P)$ is a crumbling-free module over a right CC-ring, it is nonsingular. By [16] (Lemma 2.3), $\operatorname{Ker} f$ is closed in I, and so $\operatorname{Ker} f \oplus [P/\operatorname{wsa}(P)] \cong I$. Hence $P/\operatorname{wsa}(P)$ is injective. Sufficiency follows from the fact that $\mathcal{WSS}$-co-injective modules are closed under extensions. □

Proposition 17. *A ring R is right weakly semiartinian if and only if every right R-module is $\mathcal{WSS}$-co-injective.*

Proof. Necessity is clear. For sufficiency, it is enough to show that $\mathrm{C}(M) \neq 0$ for every nonzero R-module M. Let N be a crumbling-free module. Then any submodule K of N is also crumbling-free. It follows from Proposition 10 that K is injective, therefore a direct summand of N. This shows that N is semisimple. Then we have $N = \operatorname{Soc} N \leq \mathrm{C}(N) = 0$. Hence R is right weakly semiartinian. □

A ring R is called a right *SSI-ring* if all semisimple right R-modules are injective. It is known that a ring R is a right noetherian right V-ring if and only if it is a right SSI-ring.

Theorem 5. *The following statements are equivalent for a ring R.*

1. *Every $\mathcal{WSS}$-co-injective R-module is injective;*
2. *Every weakly semiartinian R-module is injective;*
3. *R is semisimple artinian.*

Proof. $(1 \Rightarrow 2)$ and $(3 \Rightarrow 1)$ are clear.

$(2 \Rightarrow 3)$: Every semisimple module is weakly semiartinian, hence injective by assumption and so R is a right SSI-ring. Then every module crumbles by [6] (Theorem 3). Since crumbling modules are weakly semiartinian, R is semisimple artinian by assumption. □

An R-module K is called *$\mathcal{WSS}$-coprojective* if every short exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0\,,$$

of right R-modules ending with the module K is in the proper class $\mathcal{WSS}$. For an arbitrary ring R, let $\mathrm{C}(R) = \mathrm{C}(R_R)$.

Proposition 18. *Let R be a crumbling-free ring. Then $\mathcal{WSS}$-coprojective R-modules are only projective modules.*

Proof. Let M be a $\mathcal{WSS}$-coprojective R-module. Since every R-module is a factor module of a free R-module, there exist a free R-module F and an epimorphism $\psi : F \longrightarrow M$. Put $U = \operatorname{Ker}(\psi)$. Now we consider the short exact sequence $0 \longrightarrow U \xrightarrow{\iota} F \xrightarrow{\psi} M \longrightarrow 0$, where ι is the canonical injection. By the hypothesis, there exists a submodule V of F such that $F = U + V$ and $U \cap V$ is weakly semiartinian. Since $\mathrm{C}(R) = 0$, it follows from [6] (Corollary 8) that $\mathrm{C}(F) = \mathrm{C}(R)F = 0$, and so $\mathrm{C}(U \cap V) \subseteq \mathrm{C}(F) = 0$. It means that the short exact sequence $0 \longrightarrow U \xrightarrow{\iota} F \xrightarrow{\psi} M \longrightarrow 0$ splits. Hence M is projective. □

Recall that a module M is *flat* if every short exact sequence of the form,

$$0 \longrightarrow M \xrightarrow{\psi} N \longrightarrow K \longrightarrow 0\,,$$

is pure exact, that is, Im ψ is a pure submodule of N. Clearly, every projective module is flat.

Theorem 6. *Over a commutative C-ring $\mathcal{WSS}$-projective modules are flat.*

Proof. This follows from [7] (Theorem 3.9) and the fact that $\mathcal{SAS} \subseteq \mathcal{WSS}$. □

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