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Solutions to nonlinear second-order three-point boundary value problems of dynamic equations on time scales

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Abstract: In this paper, we consider existence criteria of three positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales. To show our main results, we apply the well-known Leggett–Williams fixed point theorem. Moreover, we present some results for the existence of single and multiple positive solutions for boundary value problems on time scales, by applying fixed point theorems in cones. The conditions we used in the paper are different from those in [Dogan A. On the existence of positive solutions for the one-dimensional p-Laplacian boundary value problems on time scales. Dynam Syst Appl 2015; 24: 295-304].

Key words: Time scales, dynamic equation; positive solutions; fixed point theorem

1. Introduction

The investigation of dynamic equations on time scales goes back to its discoverer Stefan Hilger [19], and it is a new field of theoretical research in mathematics. In recent years, the boundary value problems (BVPs) for dynamic equations on time scales have been noticeably studied [1–7, 11–14, 17, 18, 24–30]. The topic is inspired by the conception that dynamic equations on time scales can establish connections between continuous and discontinuous mathematics. Additionally, the work of time scales has contributed to many significant practices, e.g., in the work of insect population models, stock market, heat transfer, wound healing, and prevalent models [10, 20, 23].

In [2], Anderson studied the existence of one positive solution of the following dynamic equation on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \qquad t \in (0,T)_{\mathbb{T}},$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(T),$$

where $a \in C_{ld}(0,T)$ is nonnegative, $f : [0,\infty) \to [0,\infty)$ is continuous, $\eta \in (0,\rho(T))$, and $0 < \alpha < T/\eta$. He found some results for the existence of one positive solution of the above problem constructing the limits $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}$.

In [4], Anderson et al. studied the following BVP on time scales

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + c(t)f(u(t)) = 0, \qquad t \in (a,b)_{\mathbb{T}},$$

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$$u(a) - B_0 u^{\Delta}(\nu) = 0, , \qquad u^{\Delta}(b) = 0,$$

where $\nu \in (a,b)_{\mathbb{T}}$, $f \in C_{ld}([0,+\infty),[0,+\infty))$, $c \in C_{ld}([a,b],[0,+\infty))$, and $K_m x \leq B_0(x) \leq K_M x$ for some positive constants K_m, K_M. By using a fixed-point theorem of cone expansion and compression of functional type, they established the existence result for at least one positive solution.

In [13], Dogan investigated the following p-Laplacian BVP on time scales

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t) f(t, u(t), u^{\Delta}(t)) = 0, \qquad t \in [0,T]_{\mathbb{T}},$$

$$u(0) - B_0(u^{\Delta}(0)) = 0, \qquad u^{\Delta}(T) = 0,$$

where $\phi_p(u) = |u|^{p-2}u$, for $p > 1$. We proved the existence of triple positive solutions for the one-dimensional p-Laplacian BVP by using the Leggett–Williams fixed-point theorem.The appealing significance in our paper is that the nonlinear term f is included with first-order derivative precisely.

In [17], by using a double-fixed point theorem due to Avery et al. [8], He studied the existence of at least two positive solutions for p-Laplacian three-point BVP:

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + a(t) f(u(t)) = 0, \qquad t \in [0,T]_{\mathbb{T}},$$

satisfying the boundary conditions

$$u(0) - B_0(u^{\Delta}(\eta)) = 0, \qquad u^{\Delta}(T) = 0,$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) + B_1(u^{\Delta}(\eta)) = 0,$$

where $\eta \in (0, \rho(T))_{\mathbb{T}}$.

In [29], Sun et al. studied the eigenvalue problem of the one-dimensional p-Laplacian three-point BVP

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + \lambda h(t) f(u(t)) = 0, \qquad t \in (0,T)_{\mathbb{T}},$$

$$u(0) - \beta u^{\Delta}(0) = \gamma u^{\Delta}(\eta), \qquad u^{\Delta}(T) = 0.$$

They established some adequate assumptions for the nonexistence and existence of at least one or two positive solutions by using the Krasnosel'skii's fixed-point theorem in a cone.

In this paper, we study the following BVPs:

(1) We discuss the existence of at least three positive solutions to the following p-Laplacian BVP on time scales

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + w(t) f(u(t)) = 0, \qquad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

$$u(0) - \alpha_1 u^{\Delta}(0) = \alpha_2 u^{\Delta}(\xi), \qquad u^{\Delta}(T) = 0, \tag{1.2}$$

where $\phi_p(u)$ is p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$, for $p > 1$, with $(\phi_p)^{-1} = \phi_q$ and $1/p + 1/q = 1$. For general basic ideas and background about dynamic equations on time scales we refer the reader to [9, 10, 15].

(2) We examine the existence of one and many positive solutions to the three-point BVP on time scales

$$u^{\Delta\nabla}(t) + w(t)f(t,u(t)) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{1.3}$$

$$u(0) - \alpha_1 u^{\Delta}(0) = \alpha_2 u^{\Delta}(\xi), \qquad u^{\Delta}(T) = 0, \tag{1.4}$$

where $\alpha_1, \alpha_2 \geq 0, \quad \xi \in (0, \rho(T))$.

Motivated by the work described above, in this paper, we deal with the existence of positive solutions to BVPs (1.1),(1.2) and (1.3),(1.4). Our purpose in this work is to apply the fixed point theorem in cones. Our conceptions are analogous to those used in [13], but a little different. By applying Leggett–Williams fixed-point theorem, we have achieved novel results that are different from the earlier results. To the best of our knowledge, no one has investigated the existence of positive solutions to BVPs (1.1),(1.2) and (1.3),(1.4).

2. Preliminaries

Definition 2.1 *Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is known as a cone if it satisfies the two assumptions:*

(i) $u \in P, \ \lambda \geq 0$ *implies* $\lambda u \in P$;

(ii) $u \in P, \ -u \in P$ *implies* $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E presented by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

$\forall x, y \in P$ *and* $t \in [0,1]$.

Let $r_1, r_2, r_3 > 0$ be constants. Note that

$$P_{r_3} = \{u \in P : \|u\| < r_3\}, \qquad P(\alpha, r_1, r_2) = \{u \in P : \alpha(u) \geq r_1, \quad \|u\| \leq r_2\}.$$

Finally, we end this section by recalling a preliminary theorem, the Leggett–Williams fixed-point theorem [22], which we shall use to prove our existence results.

Theorem 2.3 *Let $F : \overline{P}_{r_3} \to \overline{P}_{r_3}$ be a completely continuous map and ψ be a nonnegative continuous concave functional on P such that $\psi(u) \leq \|u\|, \ \forall u \in \overline{P}_{r_3}$. Assume that there exist r_1, r_2, r_4, with $0 < r_4 < r_1 < r_2 \leq r_3$ such that:*

(A1) $\{u \in P(\psi, r_1, r_2) : \psi(u) > r_1\} \neq \emptyset$ *and* $\psi(Fu) > r_1$ *for all* $u \in P(\psi, r_1, r_2)$;

(A2) $\|Fu\| < r_4$ *for all* $u \in \overline{P}_{r_4}$;

(A3) $\psi(Fu) > r_1$ *for all* $u \in P(\psi, r_1, r_3)$ *with* $\|Fu\| > r_2$.

Then F has at least three fixed points u_1, u_2, u_3 satisfying

$$\|u_1\| < r_4, \qquad r_1 < \psi(u_2), \qquad \|u_3\| > r_4, \qquad \psi(u_3) < r_1.$$

3. Existence of positive solutions to BVP (1.1) and (1.2)

We will use the following assumptions in our main results:

(H1) $f : R \to (0, +\infty)$ is continuous;

(H2) $w : \mathbb{T} \to (0, +\infty)$ is left dense continuous (i.e. $w \in C_{ld}(\mathbb{T}, (0, +\infty))$), and does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$, where $C_{ld}(\mathbb{T}, (0, +\infty))$ denotes the set of all left dense continuous functions from $\mathbb{T}$ to $(0, +\infty)$, $\min_{t\in[0,T]_{\mathbb{T}}} w(t) = \phi_p(m_1)$, $\max_{t\in[0,T]_{\mathbb{T}}} w(t) = \phi_p(m_2)$, and $m_1 < m_2$;

(H3) α_1, α_2 are nonnegative constants, $\xi \in (0, \rho(T))$.

Let $u^{\Delta\nabla}(t) \le 0$, for $t \in [0, T]_{\mathbb{T}^k \cap \mathbb{T}_k}$. Then u is concave on $[0, T]_{\mathbb{T}}$.

Let $E = C^{\Delta}_{ld}([0, T]_{\mathbb{T}}, R)$ with the norm

$$\|u\| = \max\left\{\|u\|_\star, \quad \|u^\Delta\|_\star\right\},$$

where $\|u\|_\star = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$, $\|u^\Delta\|_\star = \sup_{t\in[0,T]_{\mathbb{T}^k}} |u^\Delta(t)|$; clearly E is Banach space. Choose the cone $P \subset E$ defined by

$$P = \left\{u \in E : u \text{ is nonnegative, increasing and concave on } [0, T]_{\mathbb{T}}\right\}.$$

Lemma 3.1 *Suppose that (H3) is satisfied. If $y \in C_{ld}[0, T]_{\mathbb{T}}$, then the BVP*

$$(\phi_p(u^\Delta(t)))^\nabla + y(t) = 0, \quad t \in [0, T]_{\mathbb{T}}, \tag{3.1}$$

$$u(0) - \alpha_1 u^\Delta(0) = \alpha_2 u^\Delta(\xi), \qquad u^\Delta(T) = 0, \tag{3.2}$$

has the unique solution

$$u(t) = \int_0^t \phi_q\left(\int_s^T y(\tau)\nabla\tau\right)\Delta s$$

$$+\alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right). \tag{3.3}$$

Proof Integrating (3.1) from t to T and using the second condition of (3.2), one gets

$$u^\Delta(t) = \phi_q\left(\int_t^T y(\tau)\nabla\tau\right). \tag{3.4}$$

Integrating (3.4) from 0 to t, we find

$$u(t) = u(0) + \int_0^t \phi_q\left(\int_s^T y(\tau)\nabla\tau\right)\Delta s. \tag{3.5}$$

Using the first condition of (3.2), we get

$$u(0) - \alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) = \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right).$$

Hence,

$$u(0) = \alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right). \tag{3.6}$$

Substituting (3.6) in (3.5), we find

$$\begin{aligned} u(t) = & \int_0^t \phi_q\left(\int_s^T y(\tau)\nabla\tau\right)\Delta s \\ & +\alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right). \end{aligned}$$

□

Lemma 3.2 *Let* $\alpha_1, \alpha_2 \geq 0$*. If* $y \in C_{ld}[0,T]_{\mathbb{T}}$ *and* $y \geq 0$*, then the unique solution* u *of BVP (3.1) and (3.2) satisfies*

$$u(t) \geq 0 \quad \textit{for} \quad t \in [0,T]_{\mathbb{T}}.$$

Proof In view of Lemma 3.1, one has that

$$u(0) = \alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right) \geq 0,$$

and

$$\begin{aligned} u(T) &= \int_0^T \phi_q\left(\int_s^T y(\tau)\nabla\tau\right)\Delta s \\ &\quad +\alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right) \geq 0. \end{aligned}$$

If $t \in (0,T)_{\mathbb{T}}$, we have

$$\begin{aligned} u(t) &= \int_0^t \phi_q\left(\int_s^T y(\tau)\nabla\tau\right)\Delta s \\ &\quad +\alpha_1\phi_q\left(\int_0^T y(\tau)\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T y(\tau)\nabla\tau\right) \geq 0. \end{aligned}$$

Therefore, $u(t) \geq 0, \quad t \in [0,T]_{\mathbb{T}}$. This completes the proof of the lemma. □

It is noted that $u(t)$ is a solution of the problem (1.1) and (1.2) if and only if

$$\begin{aligned} u(t) &= \int_0^t \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s \\ &\quad +\alpha_1\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right)+\alpha_2\phi_q\left(\int_\xi^T w(\tau)f(u(\tau))\nabla\tau\right). \end{aligned}$$

Define a completely continuous integral operator $A : E \to E$ by

$$\begin{aligned} (Au)(t) &= \int_0^t \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s \\ &\quad +\alpha_1\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right)+\alpha_2\phi_q\left(\int_\xi^T w(\tau)f(u(\tau))\nabla\tau\right). \end{aligned}$$

□

Lemma 3.3 $A : P \to P$.

Proof $\forall u \in P, Au \in E$ and $(Au)(t) \geq 0,\ \forall t \in [0,T]_{\mathbb{T}}$. In fact

$$(Au)^{\Delta}(t) = \phi_q\left(\int_t^T w(\tau)f(u(\tau))\nabla\tau\right) \geq 0.$$

Clearly, $(Au)^{\Delta}(t)$ is a continuous function, $(Au)^{\Delta}(t) \geq 0$, so $(Au)(t)$ is increasing on $[0,T]_{\mathbb{T}}$.

If $t \in [0,T]_{\mathbb{T}^k \cap \mathbb{T}_k}$, then $(Au)^{\Delta\nabla}(t) \leq 0$, which implies that Au is concave on $[0,T]_{\mathbb{T}}$. Thus, $Au \in P$, $A : P \to P$. □

Let $\upsilon \in \mathbb{T}$ be fixed such that $0 < \xi < \upsilon < T$. Let $\psi : P \to [0,\infty)$ be the nonnegative continuous concave functional on P. We define

$$\psi(u) = \min_{t\in[\xi,\upsilon]_{\mathbb{T}}} u(t), \quad \forall u \in P.$$

For notational convenience, we denote λ_1 and λ_2 by

$$\lambda_1 = (T+\alpha_1+\alpha_2)\phi_q\left(\int_0^T w(\tau)\nabla\tau\right), \qquad \lambda_2 = (\upsilon+\alpha_1+\alpha_2)\phi_q\left(\int_0^\xi w(\tau)\nabla\tau\right).$$

We are now ready to present growth conditions on f so that BVP (1.1) and (1.2) has at least three positive solutions.

Theorem 3.4 *Suppose that there exist nonnegative numbers $r_1, r_2, r_3,$ and r_4 such that $0 < r_4 < r_1 \leq \dfrac{m_1(\upsilon+\alpha_1+\alpha_2)}{m_2(T+\alpha_1+\alpha_2)}r_2 < r_2 \leq r_3$ and assume that f satisfies the four assumptions:*

(B1) $f(u) < \phi_p\big(r_4/\lambda_1\big)$ *for* $u \in [0, r_4]$;

(B2) $f(u) \leq \phi_p\big(r_3/\lambda_1\big)$ *for* $u \in [0, r_3]$;

(B3) $f(u) > \phi_p\big(r_1/\lambda_2\big)$ *for* $u \in [r_1, r_2]$;

(B4) $\min\limits_{u\in[0,r_3]} f(u) \times \phi_p(m_2/m_1) \int_0^{\xi} w(\tau)\nabla\tau \geq \max\limits_{u\in[0,r_3]} f(u) \times \int_0^T w(\tau)\nabla\tau.$

Then BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < r_4, \qquad r_1 < \psi(u_2), \qquad \|u_3\| > r_4, \qquad \psi(u_3) < r_1.$$

Proof Firstly, we prove that if there exists a positive number R such that $f(u) \leq \phi_p\big(R/\lambda_1\big)$ for all $0 \leq u \leq R$, then $A\overline{P}_R \subset \overline{P}_R$.

In fact, if $u \in \overline{P}_R$, then, according to Lemma 3.3, one has $A\overline{P}_R \subset P$. Additionally, if $\forall u \in \overline{P}_R$, $0 \leq u \leq R$, then one has that

$$\begin{aligned}
|Au| &= \Bigg| \int_0^t \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s \\
&\quad + \alpha_1\phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right) + \alpha_2\phi_q\left(\int_\xi^T w(\tau)f(u(\tau))\nabla\tau\right)\Bigg| \\
&\leq \int_0^T \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad + \alpha_2\phi_q\left(\int_\xi^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\leq \int_0^T \phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_0^T h(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad + \alpha_2\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&= (T+\alpha_1+\alpha_2)\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\leq \frac{R}{\lambda_1}(T+\alpha_1+\alpha_2)\phi_q\left(\int_0^T w(\tau)\nabla\tau\right) \\
&= R,
\end{aligned}$$

$$
\begin{aligned}
|(Au)^{\Delta}| &= \left|\phi_q\left(\int_t^T w(\tau)f(u(\tau))\nabla\tau\right)\right| \\
&\leq \phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\leq \phi_q\left(\int_0^T w(\tau)\nabla\tau\right)\frac{R}{\lambda_1} \\
&= \frac{R}{(T+\alpha_1+\alpha_2)} \\
&\leq R.
\end{aligned}
$$

Thus, $\|Au\| \leq R,$ which implies that $A\overline{P}_R \subset \overline{P}_R$.

Consequently, we have clarified that if (B1) and (B2) are satisfied, then $A\overline{P}_{r_4} \subset P_{r_4}$ and $A\overline{P}_{r_3} \subset \overline{P}_{r_3}$.

Secondly, we show that $\{u \in P(\psi, r_1, r_2) : \psi(u) > r_1\} \neq \emptyset$ and $\psi(Au) > r_1$ for $u \in P(\psi, r_1, r_2)$. Indeed, set $u = \dfrac{r_1+r_2}{2}$, $\|u\| = \dfrac{r_1+r_2}{2} \leq r_2$ and $\psi(u) > r_1$. Therefore, $\{u \in P(\psi, r_1, r_2) : \psi(u) > r_1\} \neq \emptyset$. In addition, $\forall u \in P(\psi, r_1, r_2),$ we get $r_1 \leq u(t) \leq r_2,$ and for $t \in [0, \upsilon]_{\mathbb{T}}$; from B3, we have

$$
\begin{aligned}
\psi(Au) &= (Au)(\upsilon) \\
&= \int_0^{\upsilon} \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad +\alpha_2\phi_q\left(\int_{\xi}^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\geq \int_0^{\upsilon} \phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad +\alpha_2\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&= (\upsilon+\alpha_1+\alpha_2)\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&> \frac{r_1}{\lambda_2}(\upsilon+\alpha_1+\alpha_2)\phi_q\left(\int_0^{\xi} w(\tau)\nabla\tau\right) \\
&= r_1.
\end{aligned}
$$

Hence, it implies that $\psi(Au) > r_1$ for $u \in P(\psi, r_1, r_2)$.

Lastly, we show that $\psi(Au) > r_1,$ for all $u \in P(\psi, r_1, r_3)$ and $\|Au\| > r_2$. If $u \in P(\psi, r_1, r_3)$ and $\|Au\| > r_2,$ then $0 \leq u(t) \leq r_3,$ $t \in [0, T]_{\mathbb{T}}$ and from condition B4, one has

$$
\phi_p\Big(\frac{m_2}{m_1}\Big)\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau \geq \int_0^T w(\tau)f(u(\tau))\nabla\tau,
$$

which can be written as

$$\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau \geq \frac{\int_0^T w(\tau)f(u(\tau))\nabla\tau}{\phi_p\left(\frac{m_2}{m_1}\right)}.$$

Therefore,

$$\begin{aligned}
\psi(Au) &= (Au)(\upsilon) \\
&= \int_0^{\upsilon} \phi_q\left(\int_s^T w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad +\alpha_2\phi_q\left(\int_{\xi}^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\geq \int_0^{\upsilon} \phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right)\Delta s + \alpha_1\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&\quad +\alpha_2\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&= (\upsilon+\alpha_1+\alpha_2)\phi_q\left(\int_0^{\xi} w(\tau)f(u(\tau))\nabla\tau\right) \\
&\geq (\upsilon+\alpha_1+\alpha_2)\phi_q\left(\frac{\int_0^T w(\tau)f(u(\tau))\nabla\tau}{\phi_p\left(\frac{m_2}{m_1}\right)}\right) \\
&= \frac{m_1(\upsilon+\alpha_1+\alpha_2)}{m_2}\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&= \frac{m_1(\upsilon+\alpha_1+\alpha_2)}{m_2(T+\alpha_1+\alpha_2)}(T+\alpha_1+\alpha_2)\phi_q\left(\int_0^T w(\tau)f(u(\tau))\nabla\tau\right) \\
&\geq \frac{m_1(\upsilon+\alpha_1+\alpha_2)}{m_2(T+\alpha_1+\alpha_2)}\|Au\| \\
&> \frac{m_1(\upsilon+\alpha_1+\alpha_2)}{m_2(T+\alpha_1+\alpha_2)}r_2 \\
&\geq r_1.
\end{aligned}$$

All the conditions of Theorem 2.1 hold. Hence BVP (1.1) and (1.2) has at least three positive solutions $u_1, u_2,$ and u_3 satisfying

$$\|u_1\| < r_4, \quad r_1 < \psi(u_2), \quad \|u_3\| > r_4, \quad \psi(u_3) < r_1.$$

□

4. Existence of three positive solutions to BVP (1.3) and (1.4)

Throughout the paper, we assume that the following assumptions hold:

(H1) $f:(0,T)\times[0,\infty)\to[0,\infty)$ is continuous;

(H2) $w:(0,T)\to[0,\infty)$ is left dense continuous such that $w(t_0)>0$ for at least one $t_0\in[\xi,T)$;

(H3) α_1,α_2 are nonnegative constants, $\xi\in(0,\rho(T))$.

Lemma 4.1 *Let* $h\in C_{ld}[0,T]_{\mathbb{T}}$. *Then the BVP*

$$u^{\Delta\nabla}(t)+h(t)=0,\quad t\in(0,T)_{\mathbb{T}}, \tag{4.1}$$

$$u(0)-\alpha_1u^{\Delta}(0)=\alpha_2u^{\Delta}(\xi),\qquad u^{\Delta}(T)=0, \tag{4.2}$$

has the unique solution

$$\begin{aligned}u(t)=&-\int_0^t(t-\tau)h(\tau)\nabla\tau+t\int_0^T h(\tau)\nabla\tau\\&+(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau-\alpha_2\int_0^{\xi}h(\tau)\nabla\tau.\end{aligned} \tag{4.3}$$

Proof By (4.1) we get

$$u(t)=-\int_0^t(t-\tau)h(\tau)\nabla\tau+C_1t+C_2.$$

By simple calculations, we can obtain

$$\begin{aligned}u(0)&=C_2, & u^{\Delta}(0)=C_1,\\ u^{\Delta}(\xi)&=-\int_0^{\xi}h(\tau)\nabla\tau+C_1, & u^{\Delta}(T)=-\int_0^T h(\tau)\nabla\tau+C_1.\end{aligned}$$

Combining this with boundary conditions (4.2), we conclude that

$$\begin{aligned}C_1&=\int_0^T h(\tau)\nabla\tau,\\ C_2&=\alpha_1\int_0^T h(\tau)\nabla\tau-\alpha_2\int_0^{\xi}h(\tau)\nabla\tau+\alpha_2\int_0^T h(\tau)\nabla\tau.\end{aligned}$$

Therefore, BVP (4.1) and (4.2) has a unique solution

$$\begin{aligned}u(t)&=-\int_0^t(t-\tau)h(\tau)\nabla\tau+t\int_0^T h(\tau)\nabla\tau\\&\quad+(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau-\alpha_2\int_0^{\xi}h(\tau)\nabla\tau.\end{aligned}$$

We can easily see that BVP $u^{\Delta\nabla}(t)=0, u(0)-\alpha_1u^{\Delta}(0)=\alpha_2u^{\Delta}(\xi)$, $u^{\Delta}(T)=0$ has only the trivial solution. As a result, u in (4.3) is the unique solution of BVP (4.1) and (4.2). □

Lemma 4.2 *Let* $\alpha_1, \alpha_2 \geq 0$. *If* $h \in C_{ld}[0,T]_{\mathbb{T}}$ *and* $h \geq 0$, *then the unique solution* u *of BVP (4.1) and (4.2) satisfies*

$$u(t) \geq 0 \quad \text{for} \quad t \in [0,T]_{\mathbb{T}}.$$

Proof In view of Lemma 4.1, one has that

$$\begin{aligned} u(0) &= (\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^\xi h(\tau)\nabla\tau \\ &\geq (\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^T h(\tau)\nabla\tau \\ &= \alpha_1\int_0^T h(\tau)\nabla\tau \geq 0 \end{aligned}$$

and

$$\begin{aligned} u(T) &= -\int_0^T (T-\tau)h(\tau)\nabla\tau + T\int_0^T h(\tau)\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^\xi h(\tau)\nabla\tau \\ &= \int_0^T \tau h(\tau)\nabla\tau + (\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^\xi h(\tau)\nabla\tau \\ &\geq \int_0^T \tau h(\tau)\nabla\tau + (\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^T h(\tau)\nabla\tau \\ &= \int_0^T \tau h(\tau)\nabla\tau + \alpha_1\int_0^T h(\tau)\nabla\tau \\ &= \int_0^T (\tau+\alpha_1)h(\tau)\nabla\tau \geq 0. \end{aligned}$$

If $t \in (0,T)_{\mathbb{T}}$, we have

$$\begin{aligned} u(t) &= -\int_0^t (t-\tau)h(\tau)\nabla\tau + t\int_0^T h(\tau)\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^\xi h(\tau)\nabla\tau \\ &= -t\int_0^t h(\tau)\nabla\tau + \int_0^t \tau h(\tau)\nabla\tau + t\int_0^T h(\tau)\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^\xi h(\tau)\nabla\tau \\ &\geq -t\int_0^T h(\tau)\nabla\tau + \int_0^t \tau h(\tau)\nabla\tau + t\int_0^T h(\tau)\nabla\tau \end{aligned}$$

$$+(\alpha_1+\alpha_2)\int_0^T h(\tau)\nabla\tau - \alpha_2\int_0^T h(\tau)\nabla\tau$$
$$= \int_0^t \tau h(\tau)\nabla\tau + \alpha_1\int_0^T h(\tau)\nabla\tau \geq 0.$$

This shows that $u(t)\geq 0$ for $t\in[0,T]_{\mathbb{T}}$ and completes the proof. □

BVP (1.3) and (1.4) has a solution $u=u(t)$ if and only if u is a fixed point of the operator equation

$$Su(t) = -\int_0^t (t-\tau)w(\tau)f(\tau,u(\tau))\nabla\tau + t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau$$
$$+(\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau - \alpha_2\int_0^\xi w(\tau)f(\tau,u(\tau))\nabla\tau. \tag{4.4}$$

Lemma 4.3 *Let $0<\xi<T$. If $h\in C_{ld}[0,T]_{\mathbb{T}}$ and $h\geq 0$, then the unique solution u of BVP (4.1) and (4.2) satisfies $\inf_{t\in[\xi,T]_{\mathbb{T}}} u(t)\geq\gamma\|u\|$, where*

$$\gamma=\frac{\xi}{T}, \qquad \|u\| = \sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|.$$

Proof Because $0\geq u^{\Delta\nabla}(t)$, we have that $u^{\Delta}(t)$ is nonincreasing. Accordingly, for $t\in[0,T]_{\mathbb{T}}$, one can write

$$u(t)-u(0) = \int_0^t u^{\Delta}(\tau)\Delta\tau \geq tu^{\Delta}(t),$$
$$u(T)-u(t) = \int_t^T u^{\Delta}(\tau)\Delta\tau \leq (T-t)u^{\Delta}(t).$$

Solving the above inequalities, we obtain

$$u(t)\geq \frac{tu(T)+(T-t)u(0)}{T} \geq \frac{t}{T}u(T) = \frac{t}{T}\|u\|.$$

Hence, it follows that

$$\inf_{t\in[\xi,T]_{\mathbb{T}}} u(t) \geq \frac{\xi}{T}\|u\|.$$

Let B be the Banach space $C_{ld}[0,T]$ with the sup norm. Describe a cone $\mathcal{P}$ in B by

$$\mathcal{P} = \Big\{u\in B : u\geq 0, \inf_{t\in[\xi,T]_{\mathbb{T}}} u(t)\geq\gamma\|u\|\Big\},$$

where $\gamma=\frac{\xi}{T}$. Clearly, $\mathcal{P}$ is a cone in B. In addition, from Lemma 4.3, $S(\mathcal{P})\subset\mathcal{P}$. We can easily see that $S:\mathcal{P}\to\mathcal{P}$ is completely continuous. □

Lemma 4.4 *([16, 21]) Let $\mathcal{P}$ be a cone in a Banach space B and D be an bounded open subset of B with $D_{\mathcal{P}} = D\cap\mathcal{P}\neq\emptyset$ and $\overline{D}_{\mathcal{P}}\neq\mathcal{P}$. Let $S:\overline{D}_{\mathcal{P}}\to\mathcal{P}$ be a completely continuous map such that $u\neq Su$ for $u\in\partial D_{\mathcal{P}}$. Let $i_{\mathcal{P}}(S,D_{\mathcal{P}})$ denote a fixed point index. Then the following results are satisfied.*

(i) *If* $\|Su\| \leq \|u\|,\ u \in \partial D_{\mathcal{P}},$ *then* $i_{\mathcal{P}}(S, D_{\mathcal{P}}) = 1.$

(ii) *If there exists* $e_1 \in \mathcal{P}\backslash\{0\}$ *such that* $u \neq Su + \lambda_1 e_1, \quad u \in \partial D_{\mathcal{P}},$ *and* $\lambda_1 > 0,$ *then* $i_{\mathcal{P}}(S, D_{\mathcal{P}}) = 0.$

(iii) *Let* U *be open in* B *such that* $\overline{U} \subset D_{\mathcal{P}}$*. If* $i_{\mathcal{P}}(S, D_{\mathcal{P}}) = 1$ *and* $i_{\mathcal{P}}(S, U_{\mathcal{P}}) = 0,$ *then* S *has a fixed point in* $D_{\mathcal{P}}\backslash\overline{U}_{\mathcal{P}}$*. The same result is satisfied if* $i_{\mathcal{P}}(S, D_{\mathcal{P}}) = 0$ *and* $i_{\mathcal{P}}(S, U_{\mathcal{P}}) = 1.$

Let $0 < r_1 < r_2,$ *and* ψ *be a nonnegative continuous concave functional on* $\mathcal{P}$*. It can be denoted that the convex sets are* $\mathcal{P}_{r_1},\ \mathcal{P}(\psi, r_1, r_2)$ *by* $\mathcal{P}_{r_1} = \{u \in \mathcal{P} : \|u\| < r_1\}$ *and* $\mathcal{P}(\psi, r_1, r_2) = \{u \in \mathcal{P} : r_1 \leq \psi(u), \|u\| \leq r_2\}.$

Define

$$\Omega_\rho = \Big\{u \in \mathcal{P} : \quad \min_{t\in[\xi,T]_{\mathbb{T}}} u(t) < \gamma\rho\Big\}.$$

Lemma 4.5 *([21]) The set* Ω_ρ *has the following properties:*

(a) Ω_ρ *is open relative to* $\mathcal{P}$*;*

(b) $\mathcal{P}_{\gamma\rho} \subset \Omega_\rho \subset \mathcal{P}_\rho$*;*

(c) $u \in \partial\Omega_\rho$ *if and only if* $\min_{t\in[\xi,T]_{\mathbb{T}}} u(t) = \gamma\rho$*;*

(d) *If* $u \in \partial\Omega_\rho,$ *then* $\gamma\rho \leq u(t) \leq \rho$ *for* $t \in [\xi, T]_{\mathbb{T}}.$

For convenience, we set

$$\frac{1}{L_1} = (T + \alpha_1 + \alpha_2)\int_0^T w(\tau)\nabla\tau, \tag{4.5}$$

$$\frac{1}{L_2} = (\xi + \alpha_1 + \alpha_2)\int_\xi^T w(\tau)\nabla\tau. \tag{4.6}$$

Also, for $\alpha \in \{0^+, \infty\},$ *we define*

$$\begin{aligned}
f^\alpha &= \limsup_{u\to\alpha}\left\{\max_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{u}\right\},\\
f_\alpha &= \liminf_{u\to\alpha}\left\{\min_{t\in[\xi,T]_{\mathbb{T}}} \frac{f(t,u)}{u}\right\},\\
f^\rho_{\gamma\rho} &= \min\left\{\min_{t\in[\xi,T]_{\mathbb{T}}} \frac{f(t,u)}{\rho} : \gamma\rho \leq u \leq \rho\right\},\\
f^\rho_0 &= \max\left\{\max_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{\rho} : 0 \leq u \leq \rho\right\}.
\end{aligned}$$

Lemma 4.6 *Assume that* f *holds the following assumptions*

$$f^\rho_0 \leq L_1 \quad \text{and} \quad u \neq Su, \quad \text{for} \quad u \in \partial\mathcal{P}_\rho, \tag{4.7}$$

then $i_{\mathcal{P}}(S, \mathcal{P}_\rho) = 1.$

Proof If $u \in \partial\mathcal{P}_\rho$, then by (4.4), (4.7), and (4.5), we get

$$\begin{aligned} Su(t) &= -\int_0^t (t-\tau)w(\tau)f(\tau,u(\tau))\nabla\tau + t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau - \alpha_2\int_0^\xi w(\tau)f(\tau,u(\tau))\nabla\tau \\ &\le t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &\le T\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &= (T+\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &\le L_1\rho(T+\alpha_1+\alpha_2)\int_0^T w(\tau)\nabla\tau = \rho = \|u\|. \end{aligned}$$

This implies that $\|Su\| \le \|u\|$ for $u \in \partial\mathcal{P}_\rho$. Hence, it follows from condition (i) of Lemma 4.4 that $i_{\mathcal{P}}(S,\mathcal{P}_\rho) = 1$. □

Lemma 4.7 *Assume that f holds the following assumptions*

$$f^\rho_{\gamma\rho} \ge \gamma L_2 \quad \text{and} \quad u \ne Su, \quad \text{for} \quad u \in \partial\Omega_\rho, \tag{4.8}$$

then $i_{\mathcal{P}}(S,\Omega_\rho) = 0$.

Proof If $e_1(t) \equiv 1$ for $t \in [0,T]_{\mathbb{T}}$; then $e_1 \in \partial\mathcal{P}$. One asserts that $u \ne Su + \lambda_1 e_1$ for $u \in \partial\Omega_\rho$ and $\lambda_1 > 0$. If this is not the case, then there exist $u_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $u_0 = Su_0 + \lambda_0 e_1$. From (4.4), Lemma 4.5 (d), and condition (4.8), we get

$$\begin{aligned} Su_0(\xi) &= -\int_0^\xi (\xi-\tau)w(\tau)f(\tau,u_0(\tau))\nabla\tau + \xi\int_0^T w(\tau)f(\tau,u_0(\tau))\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u_0(\tau))\nabla\tau - \alpha_2\int_0^\xi w(\tau)f(\tau,u_0(\tau))\nabla\tau \\ &= -\int_0^\xi (\xi-\tau)w(\tau)f(\tau,u_0(\tau))\nabla\tau + \xi\int_0^\xi w(\tau)f(\tau,u_0(\tau))\nabla\tau \\ &\quad +\xi\int_\xi^T w(\tau)f(\tau,u_0(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^\xi w(\tau)f(\tau,u_0(\tau))\nabla\tau \\ &\quad +(\alpha_1+\alpha_2)\int_\xi^T w(\tau)f(\tau,u_0(\tau))\nabla\tau - \alpha_2\int_0^\xi w(\tau)f(\tau,u_0(\tau))\nabla\tau \\ &= \int_0^\xi (\tau+\alpha_1)w(\tau)f(\tau,u_0(\tau))\nabla\tau + \int_\xi^T (\xi+\alpha_1+\alpha_2)w(\tau)f(\tau,u_0(\tau))\nabla\tau \end{aligned}$$

$$
\begin{aligned}
&\geq \quad (\xi+\alpha_1+\alpha_2)\int_{\xi}^{T} w(\tau)f(\tau,u_0(\tau))\nabla\tau \\
&\geq \quad \gamma\rho L_2(\xi+\alpha_1+\alpha_2)\int_{\xi}^{T} w(\tau)\nabla\tau,
\end{aligned}
$$

namely

$$Su_0(\xi) \geq \gamma\rho L_2(\xi+\alpha_1+\alpha_2)\int_{\xi}^{T} w(\tau)\nabla\tau. \tag{4.9}$$

In addition, in view of proof of Lemma 7 in [3], one has that

$$\min_{t\in[\xi,T]_{\mathbb{T}}} Su_0(t) = \min\{Su_0(\xi), Su_0(T)\}. \tag{4.10}$$

As a result, by (4.10), (4.9), and (4.6), we get for $t \in [\xi,T]_{\mathbb{T}}$

$$
\begin{aligned}
u_0(t) &= Su_0(t)+\lambda_0 e_1(t) \geq \min_{t\in[\xi,T]_{\mathbb{T}}} Su_0(t)+\lambda_0 \\
&= \min\{Su_0(\xi), Su_0(T)\}+\lambda_0 = Su_0(\xi)+\lambda_0 \\
&\geq \gamma\rho L_2(\xi+\alpha_1+\alpha_2)\int_{\xi}^{T} w(\tau)\nabla\tau+\lambda_0 \\
&\geq \gamma\rho+\lambda_0.
\end{aligned}
$$

Therefore we conclude that $\gamma\rho \geq \gamma\rho+\lambda_0$, which is a contradiction. Thus, by the condition (ii) of Lemma 4.4, one has that $i_{\mathcal{P}}(S,\Omega_\rho)=0$. □

Theorem 4.8 *Suppose that one of the two assumptions is satisfied:*

(B1) There exist ρ_1,ρ_2,ρ_3 *with* $0<\rho_1<\gamma\rho_2,\quad \rho_2<\rho_3$ *such that*

$$f_0^{\rho_1} \leq L_1 \quad \textit{and} \quad f_{\gamma\rho_2}^{\rho_2} \geq \gamma L_2 \quad u \neq Su \quad \textit{for} \quad u\in\partial\Omega_{\rho_2}, \quad \textit{and} \quad f_0^{\rho_3} \leq L_1.$$

(B2) There exist ρ_1,ρ_2,ρ_3 *with* $0<\rho_1<\rho_2<\gamma\rho_3$ *such that*

$$f_{\gamma\rho_1}^{\rho_1} \geq \gamma L_2 \quad \textit{and} \quad f_0^{\rho_2} \leq L_1 \quad u \neq Su \quad \textit{for} \quad u\in\partial\mathcal{P}_{\rho_2}, \quad \textit{and} \quad f_{\gamma\rho_3}^{\rho_3} \geq \gamma L_2.$$

Then BVP (1.3) and (1.4) has two positive solutions. In addition, if in (B1) $f_0^{\rho_1} \leq L_1$ *is replaced by* $f_0^{\rho_1} < L_1$, *then BVP (1.3) and (1.4) has a third positive solution* $u_3 \in \mathcal{P}_{\rho_1}$.

Proof Suppose that (B1) is satisfied. We verify that either S has a fixed point u_1 in $\partial\mathcal{P}_{\rho_1}$ or in $\Omega_{\rho_2}\backslash\overline{\mathcal{P}}_{\rho_1}$. If $u\neq Su$ for $u\in\partial\mathcal{P}_{\rho_1}\cup\partial\mathcal{P}_{\rho_3}$, in view of Lemmas 4.6 and 4.7, one has that $i_{\mathcal{P}}(S,\mathcal{P}_{\rho_1})=1$, $i_{\mathcal{P}}(S,\Omega_{\rho_2})=0$, and $i_{\mathcal{P}}(S,\mathcal{P}_{\rho_3})=1$. From Lemma 4.5 (b) and $\rho_1<\gamma\rho_2$, we get $\overline{\mathcal{P}}_{\rho_1}\subset\mathcal{P}_{\gamma\rho_2}\subset\Omega_{\rho_2}$. By the assumption (iii) of Lemma 4.4, one has that S has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{\mathcal{P}}_{\rho_1}$. Correspondingly, S has a fixed point in $\mathcal{P}_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$. The proof is analogous when (B2) is satisfied and it is omitted.This completes the proof of the theorem. □

Note that we can generalize Theorem 4.1 to find several positive solutions and it is omitted.

As consequences of Theorem 4.1, one has the next corollary.

Corollary 4.9 *Suppose that there exists $\rho > 0$ such that one of the two assumptions is satisfied:*

(C1) $0 \leq f^0 < L_1, \quad f^{\rho}_{\gamma\rho} \geq \gamma L_2, \quad u \neq Su \quad$ *for* $\quad u \in \partial\Omega_\rho, \quad$ *and* $\; 0 < f^\infty < L_1;$

(C2) $L_2 < f_0 \leq \infty, \quad f^{\rho}_{0} \leq L_1, \quad u \neq Su \quad$ *for* $\quad u \in \partial\mathcal{P}_\rho, \quad$ *and* $\; L_2 < f_\infty \leq \infty.$

Then BVP (1.3) and (1.4) has two positive solutions.

Proof We prove that (C1) means (B1). We can easily see that $0 \leq f^0 < L_1$ means that there exists $\rho_1 \in (0, \gamma\rho)$ such that $f^{\rho_1}_0 < L_1$. If $m \in (f^\infty, L_1)$, then there exists $\sigma > \rho$ such that $\max\limits_{t\in[0,T]_{\mathbb{T}}} f(t,u) \leq mu$ for $u \in [\sigma, \infty)$ because $0 \leq f^\infty < L_1$. If

$$r_2 = \max\Big\{ \max_{t\in[0,T]_{\mathbb{T}}} f(t,u) : 0 \leq u \leq \sigma \Big\}, \qquad \rho_3 > \max\Big\{ \rho, \frac{r_2}{L_1 - m} \Big\},$$

one has

$$\max_{t\in[0,T]_{\mathbb{T}}} f(t,u) \leq mu + r_2 \leq m\rho_3 + r_2 < L_1\rho_3 \;\text{ for }\; 0 \leq u \leq \rho_3,$$

which implies that $f^{\rho_3}_0 < L_1$ and (B1) is satisfied. By a similar argument, (C2) implies (B2). □

Theorem 4.10 *Suppose that one of the two assumptions is satisfied:*

(D1) There exist $\rho_1, \rho_2 > 0$ with $\rho_1 < \gamma\rho_2$ such that $f^{\rho_1}_0 \leq L_1$ and $f^{\rho_2}_{\gamma\rho_2} \geq \gamma L_2$.

(D2) There exist $\rho_1, \rho_2 > 0$ with $\rho_1 < \rho_2$ such that $f^{\rho_1}_{\gamma\rho_1} \geq \gamma L_2$ and $f^{\rho_2}_0 \leq L_1$.

Then BVP (1.3) and (1.4) has one positive solution.

As consequences of Theorem 4.2, one has the next corollary.

Corollary 4.11 *Assume that one of the next conditions is satisfied:*

(E1) $0 \leq f^0 < L_1$ *and* $L_2 < f_\infty \leq \infty;$

(E2) $0 \leq f^\infty < L_1$ *and* $L_2 < f_0 \leq \infty.$

Then BVP (1.3) and (1.4) has a positive solution.

Let $\psi : \mathcal{P} \to [0, \infty)$ be the nonnegative continuous concave functional on $\mathcal{P}$. One interprets

$$\psi(u) = \min_{t\in[\xi,T]_{\mathbb{T}}} u(t), \qquad u \in \mathcal{P}.$$

It can be noted that $\psi(u) \leq \|u\|$, for $u \in \mathcal{P}$. If L_1, L_2 are the same as in (4.5) and (4.6), then one finds the next result.

Theorem 4.12 *Assume that there exist constants $r^\star_1$ and $r^\star_4$ with $0 < r^\star_4 < r^\star_1$ such that the following assumptions hold:*

(F1) $f(t,u) < r^\star_4 L_1$ *for* $t \in [0,T]_{\mathbb{T}}, \quad 0 \leq u \leq r^\star_4;$

(F2) $f(t,u) \geq r_1^\star L_2$ *for* $t \in [\xi, T]_{\mathbb{T}}$, $r_1^\star \leq u \leq r_1^\star/\gamma$;

(F3) One of the following assumptions is satisfied;

(a) $\lim_{u\to\infty} \max_{t\in[0,T]_{\mathbb{T}}} \dfrac{f(t,u)}{u} < L_1$;

(b) There exists a number $r_3^\star > r_1^\star/\gamma$ *such that* $f(t,u) < r_3^\star L_1$ *for* $t \in [0,T]_{\mathbb{T}}$ *and* $0 \leq u \leq r_3^\star$.

Then BVP (1.3) and (1.4) has at least three positive solutions.

Proof From the description of operator S and its features, it is sufficient to clarify that the assumptions of Theorem 2.1 are satisfied.

Let $r_2^\star = r_1^\star/\gamma$. Firstly, we verify that if (a) is satisfied, then there exists a number $k^\star > r_2^\star$ such that $S : \overline{\mathcal{P}}_{k^\star} \to \mathcal{P}_{k^\star}$. If $\lim_{u\to\infty} \max_{t\in[0,T]_{\mathbb{T}}} \dfrac{f(t,u)}{u} < L_1$; then there exist $\sigma > 0$ and $\delta < L_1$ such that if $u > \sigma$, then $\max_{t\in[0,T]_{\mathbb{T}}} f(t,u)/u \leq \delta$. It implies $f(t,u) \leq \delta u$ for $t \in [0,T]_{\mathbb{T}}$ and $u > \sigma$. Let $\lambda_1 = \max\Big\{f(t,u) : t \in [0,T]_{\mathbb{T}},\ 0 \leq u \leq \sigma\Big\}$. Then we have

$$f(t,u) \leq \delta u + \lambda_1, \tag{4.11}$$

for all $t \in [0,T]_{\mathbb{T}}$, $u \geq 0$. We take

$$k^\star > \max\left\{r_2^\star, \frac{\lambda_1}{L_1 - \delta}\right\}. \tag{4.12}$$

If $u \in \overline{\mathcal{P}_k^\star}$, then by (4.4), (4.11), and (4.12), we find

$$\begin{aligned}
\|Su\| &= \max_{t\in[0,T]}\Bigg\{-\int_0^t (t-\tau)w(\tau)f(\tau,u(\tau))\nabla\tau + t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\
&\quad +(\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau - \alpha_2\int_0^\xi w(\tau)f(\tau,u(\tau))\nabla\tau\Bigg\} \\
&\leq \max_{t\in[0,T]}\left\{t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau\right\} \\
&= T\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\
&= (T+\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\
&\leq (T+\alpha_1+\alpha_2)\int_0^T w(\tau)(\delta u(\tau)+\lambda_1)\nabla\tau \\
&\leq (T+\alpha_1+\alpha_2)(\delta\|u\|+\lambda_1)\int_0^T w(\tau)\nabla\tau \\
&= \frac{\delta k^\star + \lambda_1}{L_1} < k^\star.
\end{aligned}$$

Next, we will prove that if there exists a positive number r_3 such that $f(t,u) < r_3 L_1$ for $t \in [0,T]_{\mathbb{T}},\ 0 \le u \le r_3$, then $S : \overline{\mathcal{P}}_{r_3} \to \mathcal{P}_{r_3}$. Assume that $u \in \overline{\mathcal{P}}_{r_3}$, then one has

$$\begin{aligned} \|Su\| &\le \max_{t\in[0,T]} \left\{ t\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \right\} \\ &= T\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau + (\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &= (T+\alpha_1+\alpha_2)\int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &< r_3 L_1 (T+\alpha_1+\alpha_2)\int_0^T w(\tau)\nabla\tau = r_3. \end{aligned}$$

Consequently, one has proved that if either (a) or (b) is satisfied, then there exists a number $r_3^\star$ with $r_3^\star > r_2^\star$ and $S : \overline{\mathcal{P}}_{r_3^\star} \to \mathcal{P}_{r_3^\star}$. It is also noted that from (F1) we get $S : \overline{\mathcal{P}}_{r_4^\star} \to \mathcal{P}_{r_4^\star}$.

Now, we prove that $\{u \in \mathcal{P}(\psi, r_1^\star, r_2^\star) : \psi(u) > r_1^\star\} \neq \emptyset$, and $\psi(Su) > r_1^\star$ for all $u \in \mathcal{P}(\psi, r_1^\star, r_2^\star)$.

Indeed

$$u = \frac{r_1^\star + r_2^\star}{2} \in \{u \in \mathcal{P}(\psi, r_1^\star, r_2^\star) : \psi(u) > r_1^\star\}.$$

For $u \in \mathcal{P}(\psi, r_1^\star, r_2^\star)$, we get $r_1^\star \le \min\limits_{t\in[\xi,T]_{\mathbb{T}}} u(t) \le u(t) \le r_2^\star$ for all $t \in [\xi,T]_{\mathbb{T}}$. Then, from (F2), we find that

$$\begin{aligned} \psi(Su) &= \min_{t\in[\xi,T]} Su(t) = \min\{Su(\xi), Su(T)\} = Su(\xi) \\ &\ge (\xi+\alpha_1+\alpha_2)\int_\xi^T w(\tau)f(\tau,u(\tau))\nabla\tau \\ &\ge r_1^\star L_2(\xi+\alpha_1+\alpha_2)\int_\xi^T w(\tau)\nabla\tau = r_1^\star. \end{aligned}$$

Lastly, we claim that if $u \in \mathcal{P}(\psi, r_1^\star, r_3^\star)$ and $\|Su\| > r_2^\star$, then $\psi(Su) > r_1^\star$.

Assume $u \in \mathcal{P}(\psi, r_1^\star, r_3^\star)$ and $\|Su\| > r_2^\star$, then

$$\psi(Su) = \min_{t\in[\xi,T]_{\mathbb{T}}} Su(t) \ge \gamma\|Su\| > \gamma r_2^\star = r_1^\star.$$

All the conditions of Theorem 2.1 are satisfied. Thus, BVP (1.3) and (1.4) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < r_4^\star, \quad r_1^\star < \min_{t\in[\xi,T]_{\mathbb{T}}} u_2(t), \quad \|u_3\| > r_4^\star \text{ with } \min_{t\in[\xi,T]_{\mathbb{T}}} u_3(t) < r_1^\star.$$

This completes the proof of the theorem. □

By Theorem 4.3, we notice that, when the conditions (F1),(F2), (b) of (F3) are enforced suitably on f, one can set up the existence of a random odd number of positive solutions of BVP (1.3) and (1.4).

Theorem 4.13 *If there exist constants*

$$0 < r^{\star}_{4_1} < r^{\star}_{1_1} < \frac{r^{\star}_{1_1}}{\gamma} < r^{\star}_{4_2} < r^{\star}_{1_2} < \frac{r^{\star}_{1_2}}{\gamma} < r^{\star}_{4_3} < \cdots < r^{\star}_{4_n}, \quad n \in \mathbb{N},$$

such that the two assumptions are satisfied:

(G1) $f(t,u) < r^{\star}_{4_i} L_1$ *for* $t \in [0,T]_{\mathbb{T}}$, $u \in [0, r^{\star}_{4_i}]$;

(G2) $f(t,u) \geq r^{\star}_{1_i} L_2$ *for* $t \in [\xi,T]_{\mathbb{T}}$, $u \in [r^{\star}_{1_i}, r^{\star}_{1_i}/\gamma]$.

Then, BVP (1.3) and (1.4) has at least $2n-1$ *positive solutions.*

Proof Let $n = 1$, then it is instant by assumption (G1) that $S : \overline{\mathcal{P}}_{r^{\star}_{4_1}} \to \mathcal{P}_{r^{\star}_{4_1}} \subset \overline{\mathcal{P}}_{r^{\star}_{4_1}}$, which implies that S has at least one fixed point $u_1 \in \overline{\mathcal{P}}_{r^{\star}_{4_1}}$ by the Schauder fixed-point theorem. Let $n = 2$, then it is obvious that Theorem 4.3 is satisfied ($r_{3_1} = r^{\star}_{4_2}$). It now follows that BVP (1.3) and (1.4) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_1\| < r^{\star}_{4_1}, \quad r^{\star}_{1_1} < \min_{t\in[\xi,T]_{\mathbb{T}}} u_2(t), \quad \|u_3\| > r^{\star}_{4_1} \quad \text{with} \quad \min_{t\in[\xi,T]_{\mathbb{T}}} u_3(t) < r^{\star}_{1_1}.$$

Following this procedure, one can conclude the proof by induction. □

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