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Cihan Ciftci, Hatice Sinem Sas Cayci, Mehmet Tarik Atay, Batuhan Toker, Berkay Guncan, and Afsin Talha Yildirim

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The Numerical Solutions for Stiff Ordinary Differential Equations by Using Interpolated Variational Iteration Method with Comparison to Exact Solutions

Cihan Ciftci [2] , Hatice Sinem Sas Cayci [1], Mehmet Tarik Atay [1,a],
Batuhan Toker[1] , Berkay Guncan[1] , Afsin Talha Yildirim[1]

[1] *Engineering Faculty, Mechanical Engineering Department ,Abdullah Gül University , Kayseri, Turkey.*
[2] *Engineering Faculty, Civil Engineering Department ,Abdullah Gül University , Kayseri, Turkey.*

[a)] Corresponding author mehmettarik.atay@agu.edu.tr

Abstract. Recently proposed Interpolated Variational Iteration Method (IVIM) is used to find numerical solutions of stiff ordinary differential equations for both linear and nonlinear problems. The examples are given to illustrate the accuracy and effectiveness of IVIM method and IVIM results are compared with exact results. In recent analytical approximate methods based studies related to stiff ordinary differential equations, problems were solved by Adomian Decomposition Method and VIM and Homotopy Perturbation Method, Homotopy Analysis Method etc. In this study comparisons with exact solutions reveal that the Interpolated Variational Iteration Method (IVIM) is easy to implement. In fact, this method is promising methods for various systems of linear and nonlinear stiff ordinary differential equations as an initial value problem. Furthermore, IVIM is giving very satisfactory solutions when compared to exact solutions for nonlinear cases depending on the stiffness ratio of the stiff system to be solved.

Introduction

Stiff systems of ordinary differential equations (ODEs) are generally used to formulate and represent modeling of various real-world problems [1]. Obtaining the solution of stiff system of ODEs is crucial for explaining the specific behavior of the physical system itself, and many numerical and analytical methods have been proposed and formulated [1–10] for this purpose. For more detailed explanations of stiff ODE problems, some previous studies can be useful in this field [1–10]. Examples are presented to illustrate the efficiency of the various proposed approaches.

In this paper, Interpolated Variation Iteration Method (IVIM) is used to obtain numerical solutions of a stiff system of ODEs. This recently proposed numerical scheme is based upon Variational Iteration Method which is combined with liner interpolation functions [14]. The Variation Iteration Method was first proposed by He [11–12] and was successfully applied to autonomous ordinary differential equations by He [12], nonlinear polycrystalline solids, nonlinear partial differential equations, and other fields [11–13].

The aim of this study is to show the merits of using IVIM in solving some stiff systems of ordinary differential equations. The Variation Iteration Method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. There is no requirement for linearization or discretization and, hence, large computational work and round-off errors are avoided. The availability of computer symbolic packages gives a mathematical tool to perform some complicated manipulations and to carry out some modifications on a method for a specific problem easily. On the other hand, VIM procedure produces sequence of functions that may converge to the solution of the problem at hand with a few iterative steps involving computation of definite integral [14].

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Generally, this definite integral contain nonlinear functions and terms in them as an integrand function which create difficulties for integral calculation after few iterations [14]. The results of the Interpolated Variation Iteration Method (IVIM) are compared with the exact solutions of the problems to show the efficiency of the newly proposed IVIM method.

Variation Iteration Method (VIM)

In 1978, Inokuti et al. [16] introduced a general Lagrange Multiplier method to solve nonlinear problems. In this method the solution of a mathematical problem with a linearization assumption is used as an initial approximation or trial function. To illustrate this method, consider the following form of a differential equation:

$$Lu(t) + Nu(t) = g(t) \tag{1}$$

Where L and N are linear and nonlinear operators respectively and $g(t)$ is an inhomogenous term. By using initial guess $u_0(t)$ to construct the iteration formula, we can write the VIM formulation as follows:

$$u_{m+1}(t) = u_m(t) + \int_{t_0}^{t} \lambda(s,t)\,(\,Lu_m(s) + Nu_m(s) - g(s))\,ds \quad , \quad m = 0,1,2,3,\ldots \tag{2}$$

Where λ is a Lagrange Multiplier. (for more details, see [11, 12, 13]).

Interpolated Variational Iteration Method (IVIM)

Consider the one-dimensional initial value problem:

$$u'(t) = f\big(t, u(t)\big),\ t \,\epsilon\, [a, T],\ u(a) = u_a \tag{3}$$

For clarity, let us assume that $u_a = 0$; otherwise, by utilizing a change of variable $\tilde{u} = u - u_a$ toformulate $\tilde{u}(a) = 0$. Hence, Eq. (2) can be rewritten:

$$u_{m+1}(t) = u_m(t)\ \int_a^t \lambda(s,t)(u'_m(s) - f(s, u_m(s)))ds,\ m = 0, 1, 2, \ldots, \tag{4}$$

here $u_0(t)$ satisfies the initial condition $u_0(a) = 0$. By using integrating by parts, Eq. (4) can be formulated as given in [14]:

$$u_{m+1}(t) = G_m(t)\ \int_a^t H_m(s,t)\,ds, \tag{5}$$

where,

$$G_m(t) = \big(1 + \lambda(t,t)\big)u_m(t) - \lambda(a,t)u_m(a), \quad \text{with} \tag{6}$$

$$H_m(s,t) = \frac{\partial \lambda}{\partial s}(s,t)u_m(s) + \lambda(s,t)f\big(s, u_m(s)\big). \tag{7}$$

In order to formulate formulate IVIM (Interpolated Variational Iteration Method) approach as given in [14], consider a natural number n and discretize the interval $[a, T]$ into $n - 1$ subintervals with a step size of $h = (T - a)/(n - 1)$ and grid points:

$$t_i = a + (i - 1)h,\ \ i = 1, 2, \ldots, n \tag{8}$$

Now, we define the B-spline basic functions (see [30]) of first order on the nodal points t_i, on [*a, T*].

The main idea of IVIM is to compute a piecewise linear interpolation $u_{m+1}^{(h)}$ of u_{m+1} instead of computing $u_{m+1}(t)$. Further details of convergence behavior of IVIM and its theoretical base can be found in the work of. Salkuyeh, D. K and Tavakoli, A. [14]

Stiffness of a System of Ordinary Differential Equations

For a given system of ordinary differential equations, stiffness means a big difference in the time scales of the components in the vector solution. Some of well-known numerical methods and their procedures which are quite satisfactory in general can work unsatisfactorily on stiff problems. For a general formulation for stiff problems,

$$y' = f\,(x, y)\,, \quad y\,(x_0) = y_0\,, \quad a \le x \le b \tag{9}$$

Where $f\,(x, \boldsymbol{y})$ is defined and continuous in a region D ⊂ [a, b], and that is either singular or stiff or both. The definition of stiffness is given in [8] and [9] as follows:

The Initial Value Problem of (6) is said to be stiff over the finite interval for every x ∈ [a,b], the eigenvalues $\{\lambda_s(x), s = 1,2,3, \dots, m\}$ of the Jacobian matrix satisfy the following conditions:

$$(1)\ Re\ (\lambda\_s\,(x)) < 0\ ,\ s = 1, 2, 3, \dots, m$$

$$(2)\ stiffness\ ratio\ \ R = \llbracket max\ \rrbracket\ \|Re(\lambda\_s\,(x)\ \|/\ \llbracket min\ \rrbracket\ \|Re(\lambda\_s\,(x)\ \| \gg 1,\ s = 1,2,3, \dots., m \tag{10}$$

where λ_s are the eigenvalues of the Jacobian of the system.

Applications

In this section Interpolated Variational Iteration Method (IVIM) is applied to various stiff system of differential equations.

5.1.1. The stiff differential equation of first order can be considered as a first example of this study elaborated previously in [19]. The problem is formulated as follows:

$$y'(t) = -200[y(t) - F(t)] + F'(t)\ ,\ F(t) = 10 - (\,10 + t)e^{-t} \quad with\ \ y(0) = 10\,. \tag{11}$$

The Exact Solution is given as $y_E(\,t) = F(t) + 10e^{-200t}$.

Fig. 1 shows the exact solution and IVIM solution by defining m=5 as fifth iteration of IVIM and n=10000. The solutions computed by IVIM converges to the exact solution.

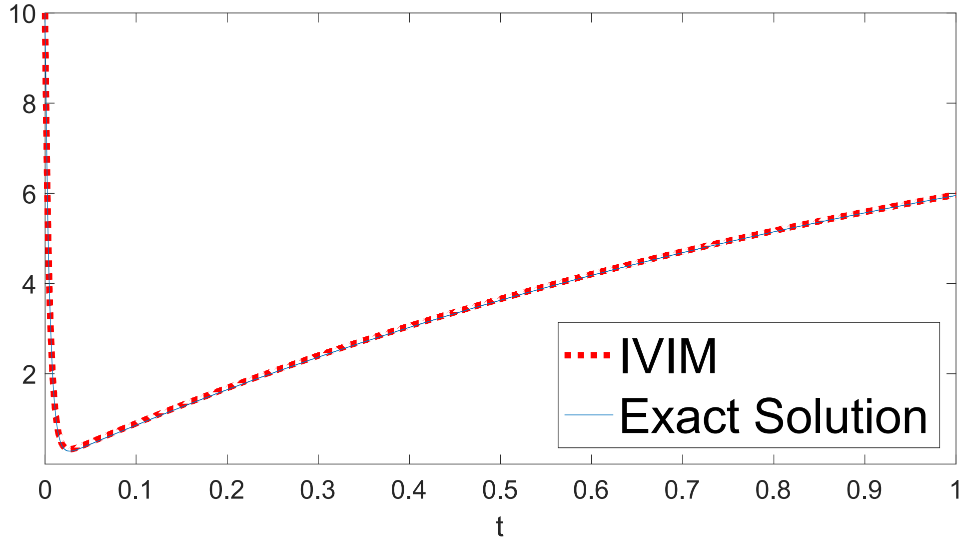

FIGURE 1. Exact solution and the approximate solutions obtained by the IVIM for m=5 and n=10000 by IVIM

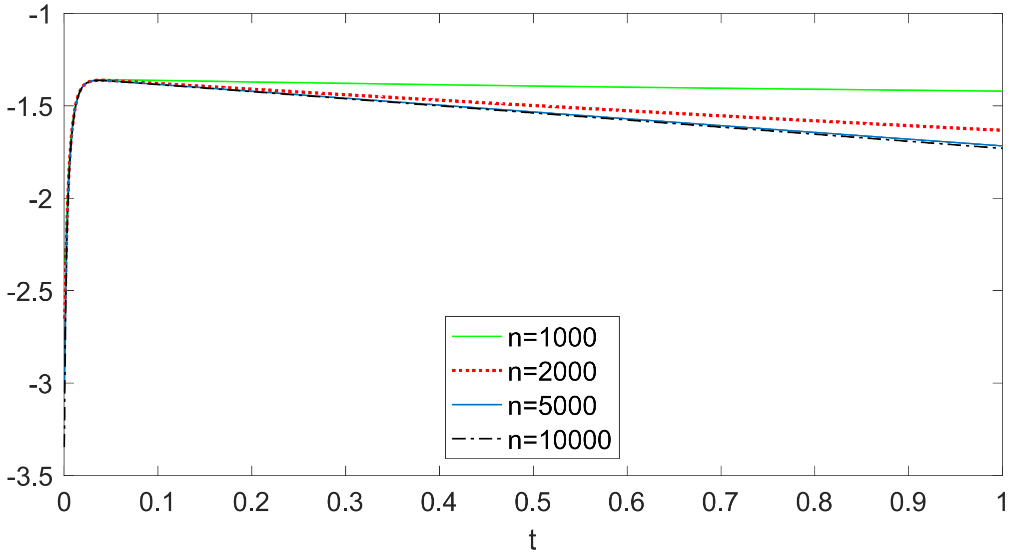

FIGURE 2. $\log_{10}$ of the absolute error of the solutions obtained by the IVIM for m=1 and n=1000,2000,5000,10000

Then, we set m=1 again and n=1000, 2000, 5000, 10000 to see relationship between step size and absolute error, which is shown in Fig. 2.

5.1.2. The stiff differential equation of first order can be considered as a second example of this study elaborated previously in [20]. The problem is formulated as follows:

$$y'(t) = 5e^{5t}\,(y(t) - t)^2 + 1 \quad with \quad y(0) = -1\,. \tag{12}$$

With the Exact Solution is given as $y_E(t) = t - e^{-5t}$.

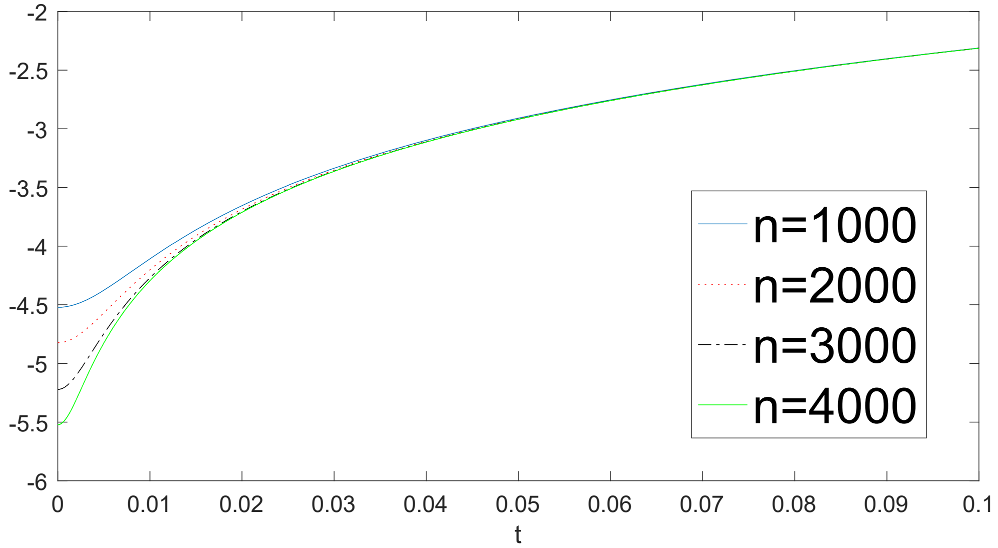

FIGURE 3. log_{10} of the absolute error of the solutions obtained by the IVIM for m=1 and n=1000,2000,5000,10000

Fig. 3 represents log_{10} of the absolute error of the solutions computed by IVIM for m=1. The IVIM results are suitable to the corresponding exact solution. However, for all four values of n, log_{10} of the absolute error of the solutions obtained by the IVIM is represented in Fig. 3.

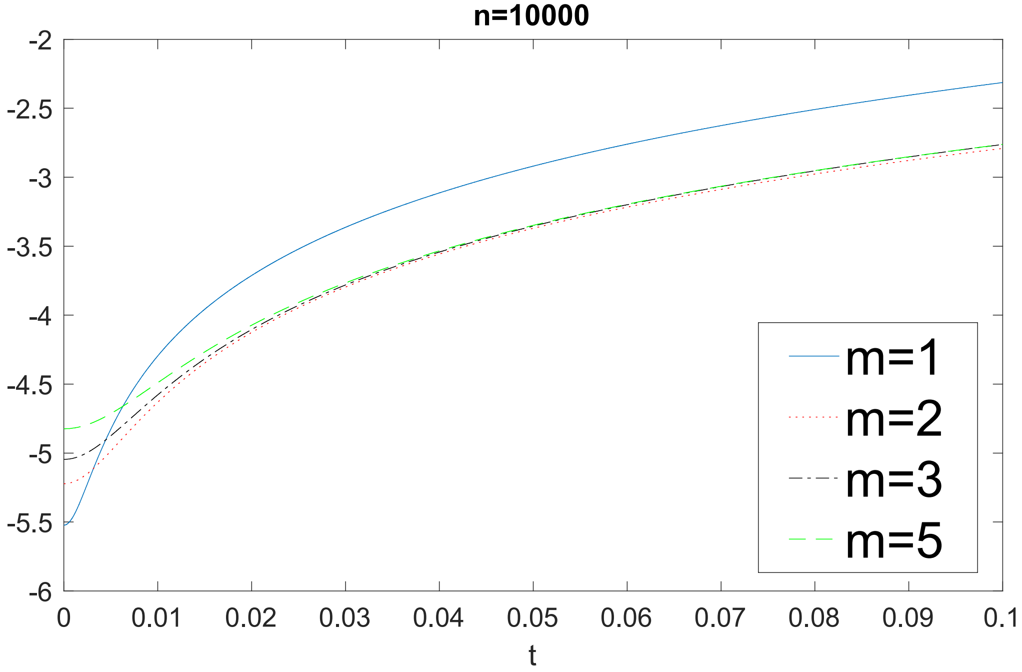

FIGURE 4. log_{10} of the absolute error of the solutions obtained by the IVIM for n=10000 and m=1,2,3, 5

Conclusion

In this study, recently proposed IVIM (Interpolated Variational Iteration Method) is applied for solving one dimensional stiff type initial value problems. The numerical results showed that IVIM is effective for solving nonlinear problems as it was elaborated in example questions. Based on IVIM numerical formulation nature, this method can be used to solve system of ODEs especially for nonlinear ODEs and systems of them. Using other types of basic functions for splines can be used for further numerical investigations to obtain more accurate results of questions at hand.

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