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RINGS WITH VARIATIONS OF FLAT COVERS

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Abstract. We introduce flat e-covers of modules and define e-perfect rings as a generalization of perfect rings. We prove that a ring is right perfect if and only if it is semilocal and right e-perfect which generalizes a result due to N. Ding and J. Chen. Moreover, in the light of the fact that a ring R is right perfect if and only if flat covers of any R-module are projective covers, we study on the rings over which flat covers of modules are (generalized) locally projective covers, and obtain some characterizations of (semi) perfect, A-perfect and B-perfect rings.

1. Introduction

Throughout this study, all rings are associative with identity and all modules are unital right modules unless indicated otherwise. For a ring R, $J(R)$ and $\mathrm{Soc}(R)$ denote the Jacobson radical of R and the socle of R_R, respectively. We use the notation $N \leq M$ for a submodule N of a module M.

An epimorphism $\alpha : P \longrightarrow M$ is said to be a *projective cover* of M if P is projective and $\mathrm{Ker}\,\alpha \ll P$, where a submodule N of a module M is called *small* in M and denoted by $N \ll M$, if $N + L \neq M$ for every proper submodule L of M. Bass calls a ring R *right (semi) perfect* if every (finitely generated) R-module has a projective cover (see [6]).

Let R be a ring, X be a class of R-modules which is closed under isomorphic copies and M be an R-module. Following [13], a homomorphism $\alpha : G \longrightarrow M$ with $G \in X$ is said to be an X*-cover* of the module M if it has the following properties:

(1) for every homomorphism $\beta : H \longrightarrow M$ with $H \in X$, there is a homomorphism $\gamma : H \longrightarrow G$ such that $\alpha\gamma = \beta$,

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(2) if ξ is an endomorphism of G with $\alpha\xi = \alpha$, then ξ is an automorphism.

Let R be a ring and X be the class of all flat R-modules. Then an X-cover $\alpha : G \longrightarrow M$ of M is said to be a *flat cover* of the module M. If X is the class of all projective modules, it follows from [19, Theorem 1.2.12] that a homomorphism $\alpha : G \longrightarrow M$ with $G \in X$ is an X-cover if and only if $\alpha : G \longrightarrow M$ is a projective cover of the module M. In [7, Theorem 3], Bican et. al. proved that every module has a flat cover. In general a flat cover of a module M need not be a projective cover of the module M. Even if a module has a projective cover, that cover need not be a flat cover. Let $\alpha : F \longrightarrow M$ be a flat cover of M. If F is projective, then the flat cover $\alpha : F \longrightarrow M$ is a projective cover of M. It follows from this fact and [19, Theorem 1.2.12] that a ring R is right perfect if and only if flat covers of any R-module are projective if and only if flat covers of any R-module M are projective covers of M. In the light of this fact, right A-perfect rings and right B-perfect rings are introduced in [2] and [8], respectively. It is shown that a ring R is right A-perfect (right B-perfect resp.) if and only if flat covers of cyclic (simple resp.) modules are projective. Note that the class of semiperfect rings properly contains the class of right B-perfect rings.

There are two different approaches to generalize right perfect rings one of which has already been mentioned. The other one is obtained by weakening the conditions in the definition of a projective cover. In [1], the term projective is replaced with flat to define flat covers of modules (called flat B-covers in [11]). Right G-perfect rings are defined as rings whose modules have flat B-covers. Another generalization is obtained by the use of δ-small submodules instead of small submodules. Here a submodule N of a module M is said to be δ*-small* in M and denoted by $N \ll_\delta M$, if whenever $N + K = M$ for some submodule K of M with M/K singular, then $K = M$. In [22] ([4] resp.) a ring R is called *right* δ*-perfect* (*right generalized* δ*-perfect* (shortly, right G-δ-perfect) resp.) if every right R-module has a projective (flat resp.) δ-cover.

In section 2, we call an epimorphism $f : K \longrightarrow M$ such that K is flat and f has e-small kernel a flat e-cover of the module M, where a submodule N of a module M is said to be *e-small* in M and denoted by $N \ll_e M$, if whenever $N + K = M$ for an essential submodule K of M, then $K = M$. Then the following straightforward implications on submodules hold:

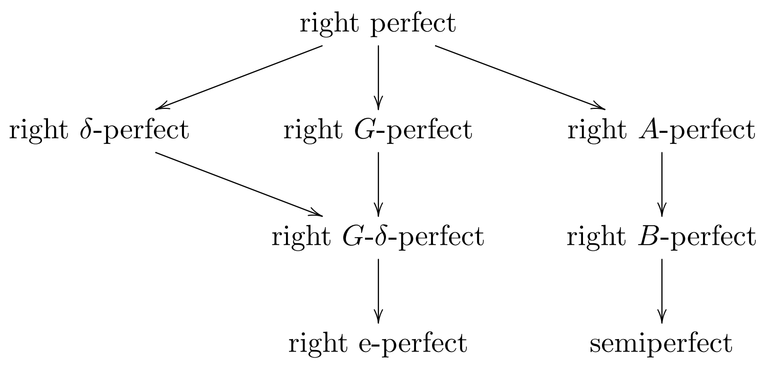
$$\text{small} \Longrightarrow \delta\text{-small} \Longrightarrow \text{e-small}$$

We call a ring R *right e-perfect* if every right R-module has a flat e-cover. By definition of e-covers, this is a generalization of right G-δ-perfect rings. In section 3, we show among other results that the class of right e-perfect rings is closed under finite direct products, factor rings and Morita equivalence. Every semilocal commutative ring whose simple modules have flat e-covers is semiperfect. We prove that a ring is right perfect if and only if it is semilocal and right e-perfect. This generalizes a result due to N. Ding and J. Chen (see [12, Theorem 4]).

It is also natural to study the rings over which flat covers of modules are (generalized) locally projective covers. A module M is called *locally projective* in case whenever $g : N \longrightarrow K$ is an epimorphism and $f : M \longrightarrow K$ is a homomorphism, then for every finitely generated submodule M_0 of M there exists a homomorphism $h : M \longrightarrow N$ such that $gh|_{M_0} = f|_{M_0}$. An epimorphism $\alpha : P \longrightarrow M$ is said to be a (*generalized*) *locally projective cover* of M if P is a locally projective module and ($\operatorname{Ker}\alpha \subseteq \operatorname{Rad}(P)$) $\operatorname{Ker}\alpha \ll P$, where $\operatorname{Rad}(P)$ is the Jacobson radical of P.

In section 4, we introduce a flat (generalized) locally projective cover (shortly, an ($FGLP$) FLP-cover) of a module. We show that a ring R is semilocal and flat covers of simple R-modules are locally projective if and only if every simple R-module has an $FGLP$-cover. We prove that a ring R is right A-perfect (right B-perfect resp.) if and only if every cyclic (simple resp.) R-module has an FLP-cover. Furthermore, we prove that a ring R is right perfect if every semisimple R-module has an FLP-cover.

Let us note that the following implications on the classes of rings are all strict:



2. Flat e-covers of modules

In [21], an epimorphism $f : K \longrightarrow M$ is called *e-small* if $\operatorname{Ker} f \ll_e K$. Using this definition, now we give the notion of e-cover as a generalization of δ-cover.

Definition 2.1. *Let M be a module. A module K together with an e-small epimorphism $f : K \longrightarrow M$ is called an e-cover of M.*

Let us begin with some basic properties of e-small modules and e-covers. Some of the proofs for the following results are omitted for being straightforward.

Lemma 2.2. *If $f_i : K_i \longrightarrow M_i$ is an e-cover of M_i for every $i = 1, 2, \ldots, n$, then $\bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} K_i \longrightarrow \bigoplus_{i=1}^{n} M_i$ is an e-cover of $\bigoplus_{i=1}^{n} M_i$.*

Whereas are their predecessors, the class of e-covers is not closed under composition.

Example 2.3. *Let $f : \mathbb{Z}_{24} \longrightarrow \mathbb{Z}_6$ and $g : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_3$ be epimorphisms given by $f(\tilde{a}) = \overline{a}$ and $g(\overline{b}) = \hat{b}$. Then,* $\operatorname{Ker} f = 6\mathbb{Z}_{24} \ll_e \mathbb{Z}_{24}$ *by* [21, Example 2.13] *and* $\operatorname{Ker} g = 3\mathbb{Z}_6 \ll_e \mathbb{Z}_6$ *by* [21, Example 2.2] *so that f and g are e-covers. However, gf is not an e-cover, since* $\operatorname{Ker} gf = 3\mathbb{Z}_{24}$ *is not e-small in $\mathbb{Z}_{24}$ by* [21, Example 2.13].

Lemma 2.4. *Let M be a module and L be a direct summand of M. If a submodule N of M is e-small in M, then $N \cap L$ is e-small in L.*

Proposition 2.5. *Let P be a projective module and K be a submodule of P. If P/K has an e-cover, then there is an e-cover $\phi : P/L \longrightarrow P/K$ of P/K for some submodule $L \leq K$ with* $\operatorname{Ker} \phi = K/L$.

Proof. Let $\alpha : F \longrightarrow P/K$ be an e-cover of P/K and $\pi : P \longrightarrow P/K$ be the canonical epimorphism. Projectivity of P guarantees the existence of a homomorphism $\beta : P \longrightarrow F$ satisfying $\alpha\beta = \pi$. Since $\operatorname{Im} \beta + \operatorname{Ker} \alpha = F$ and $\operatorname{Ker} \alpha \ll_e F$, it follows from [21, Proposition 2.3] that $\operatorname{Im} \beta$ is a direct summand of F and so $\operatorname{Ker} \alpha \cap \operatorname{Im} \beta \ll_e \operatorname{Im} \beta$ by Lemma 2.4. Let $\gamma = \alpha|_{\operatorname{Im} \beta}$, $L = \operatorname{Ker} \beta$ and $f : P/L \longrightarrow \operatorname{Im} \beta$ be the isomorphism induced by β. Then for the epimorphism $\phi = \gamma f : P/L \longrightarrow P/K$, we have $\operatorname{Ker} \phi = K/L = f^{-1}(\operatorname{Ker} \gamma) \ll_e P/L$ by [21, Proposition 2.5]. □

The following result is immediate from the proof for Proposition 2.5.

Corollary 2.6. *If a finitely generated (cyclic resp.) module M has an e-cover K, then there is a finitely generated (cyclic resp.) direct summand of K which is an e-cover of M.*

Let M be a module. Recall that the Jacobson radical $\mathrm{Rad}(M)$ of M is the sum of all small submodules of M. Following [22, Lemma 1.5 (1)] and [21, Theorem 2.10], $\delta(M)$ and $\mathrm{Rad}_e(M)$ stand for the sum of all δ-small submodules of M and the sum of all e-small submodules of M, respectively. Clearly, we have that $\mathrm{Rad}(M) \subseteq \delta(M) \subseteq \mathrm{Rad}_e(M)$ and $\mathrm{Soc}(M) \subseteq \mathrm{Rad}_e(M)$. For a ring R, let $\delta(R) = \delta(R_R)$.

The following result is interesting.

Lemma 2.7. *Let M be a module. Then $M \ll_e M$ if and only if M is semisimple.*

Proof. Let $M \ll_e M$ and N be any submodule of M. Clearly, $M = N + M$ and so, by [21, Proposition 2.3], there exists a semisimple submodule K of M such that $M = N \oplus K$. Therefore N is a direct summand of M. It means that M is semisimple. The converse is clear. □

Lemma 2.8. *Let M be a module. If* $\mathrm{Rad}(M) \ll_e M$*, then* $\mathrm{Rad}(M) \ll M$.

Proof. Let $\mathrm{Rad}(M) + N = M$ for some proper submodule N of M. By [21, Proposition 2.3], we can write $M = N \oplus (\bigoplus_{i\in I} S_i)$ for some index set I, where S_i is simple for every $i \in I$. Since $N \neq M$, I is nonempty. Then for $i_0 \in I$, $K = N \oplus (\bigoplus_{\substack{i\in I \\ i\neq i_0}} S_i)$ is a maximal submodule of M and we have $M = \mathrm{Rad}(M) + N \leq K$, which is a contradiction. □

Corollary 2.9. *Let M be a module. If* $\mathrm{Rad}_e(M) \ll_e M$*, then* $\mathrm{Rad}(M) \ll M$.

Proof. It is a consequence of Lemma 2.8 and [21, Proposition 2.5 (1)(a)]. □

In [10], a module M is said to be an *md-module* if every maximal submodule of M is a direct summand of M. The basic properties of these modules are given in the same paper. In particular, it is shown in [10, Theorem 6.11] that over a Dedekind domain a module M is an md-module if and only if $M/T(M)$ is injective and $T(M) = M_1 \oplus M_2$ with M_1 semisimple and M_2 injective, where $T(M)$ is the torsion submodule of M. Now we give an analogous characterization of this fact for md-modules over any ring. It is well known that over a Dedekind domain a module M is injective if and only if $M = \mathrm{Rad}(M)$.

Proposition 2.10. *Let M be a module. Then M is an md-module if and only if $M = \mathrm{Rad}_e(M)$.*

Proof. It is clear since $\mathrm{Rad}_e(M)$ is the intersection of all essential submodules of M. □

Observe from Proposition 2.10 and [10, Corollary 3.4] that a ring R is semisimple if and only if for every R-module M, $M = \mathrm{Rad}_e(M)$.

Definition 2.11. *Let M be a module. A flat module F together with an e-cover $f : F \longrightarrow M$ is called* a flat e-cover *of M.*

The property of having flat e-covers is not inherited by submodules. As an example, for the canonical epimorphism $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, we have $\mathbb{Q}$ is flat and $\mathbb{Z} \ll_e \mathbb{Q}$. Therefore $\mathbb{Q}$ is a flat e-cover of $\mathbb{Q}/\mathbb{Z}$, but its direct summand $\mathbb{Z}_{p^\infty}$ does not have a flat e-cover. However, we have it in a special case. The proof for the following result is similar to that for [1, Proposition 3.11], therefore it is omitted.

Proposition 2.12. *Let R be a ring such that $\mathrm{Rad}_e(M) = M\delta(R) \ll_e M$ for every flat R-module M, L be an R-module and K be a submodule of L with L/K flat. If L has a flat e-cover, then so does K.*

For a projective module M, we have $\delta(M) = \mathrm{Rad}_e(M)$ by [22, Lemma 1.5]. Furthermore, δ-small submodules of M are exactly e-small submodules of M. Therefore, in case F is projective, the definition given above coincides with the one given for a projective δ-cover. Indeed, we have similar results concerning locally projective covers. Note that projective modules are locally projective and locally projective modules are flat.

Proposition 2.13. *Let P be a locally projective module and K be a submodule of P. Then the following hold.*

(1) $\delta(P) = P\delta(R) = \mathrm{Rad}_e(P)$.
(2) *If $K \ll_e P$, then $K \ll_\delta P$.*

Proof. (1) can be proved using similar arguments as in [4, Proposition 2.8] with the help of [5, Proposition 6] and [21, Proposition 2.5(2)].

The proof for (2) follows from the fact that simple locally projective modules are projective by [5, Proposition 10] for being finitely generated. □

Remark 2.14. *Following Proposition 2.13, there is no need to distinguish between an e-cover and a δ-cover of a module in case it is locally projective. Moreover, there is no need to define a generalized version of* *an e-cover in this case. Therefore, we stick with the terms locally projective δ-cover and generalized locally projective δ-cover as in* [4]. *We use $\delta(R)$ instead of $\mathrm{Rad}_e(R_R)$ for the same reason.*

We denote by $\mathrm{Soc}_s(M)$ the sum of all simple submodules of a module M that are small in M as in [21]. Using similar arguments as in the proof for Proposition 2.13 we obtain the following result.

Proposition 2.15. *Let P be a locally projective module. Then $P\,\mathrm{Soc}_s(R_R) = \mathrm{Soc}_s(P)$.*

3. E-perfect rings

In this section, we define e-perfect rings using flat e-covers and obtain some properties of such rings. In particular, we prove that a semilocal ring is right e-perfect if and only if it is right perfect. This generalizes a result due to N. Ding and J. Chen (see [12, Theorem 4]).

Definition 3.1. *A ring R is called* right e-perfect *if every right R-module has a flat e-cover. Left e-perfect rings are defined in a similar manner. R is said to be e-perfect if it is both a right and a left e-perfect ring.*

We begin with some examples of right e-perfect rings.

Example 3.2.

(1) *Every flat module is a flat e-cover of itself. Therefore, every regular ring is right e-perfect.*
(2) *A right G-δ-perfect rings is right e-perfect, since δ-covers are e-covers.*

Next we give some general closure properties of right e-perfect rings as follows.

Proposition 3.3.

(1) *Every finite direct product of right e-perfect rings is right e-perfect.*
(2) *Every factor ring of a right e-perfect ring is right e-perfect.*
(3) *The property of a ring being right e-perfect is preserved under Morita equivalence.*

Proof. (1) and (2) follows from similar arguments used in proving [1, Proposition 2.6]. The proof for (3) is similar to that for [4, Proposition 3.7(1)] with the help of [21, Proposition 3.2(1)]. □

By Proposition 2.13 it is possible to state related versions of some results from [4]. Therefore we omit their proofs. Recall that a subset I of a ring R is called *right T-nilpotent* if for every sequence $a_1, a_2, ...$ of elements in I, there is a $k \in \mathbb{Z}^+$ with $a_k \cdots a_2 a_1 = 0$ (see [3, p. 314]).

Theorem 3.4. *The following statements are equivalent for a ring R.*

(1) $J(R/\operatorname{Soc}(R_R))$ *is right T-nilpotent.*
(2) $\operatorname{Rad}_e(F) \ll_e F$ *for every (non-semisimple) projective module F.*
(3) $\operatorname{Rad}_e(F) \ll_e F$ *for every countably generated (non-semisimple) free module F.*

Theorem 3.5. *Let R be a right e-perfect ring over which every cyclic flat module is projective. Then $R/\delta(R)$ is regular.*

It is well known that every simple flat module over a semiperfect ring is projective. Under the condition "every simple flat module is projective", a result similar to [17, Theorem 3.8] can be given using flat e-covers. Recall from Lomp [17] that a module M is said to be *semilocal* if $M/\operatorname{Rad}(M)$ is semisimple and a ring R is said to be *semilocal* if it is semilocal as a right (left) module over itself.

Theorem 3.6. *Let R be a ring over which every simple flat module is projective. Then R is semiperfect if and only if it is semilocal and every simple R-module has a flat e-cover.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Let $R/J(R) = \bigoplus_{i=1}^{n} M_i$ with M_i simple for $1 \leq i \leq n$. Note that if M is a simple module then $M \cong M_i$ for some $1 \leq i \leq n$. By assumption, there is a flat e-cover $f_i : F_i \longrightarrow M_i$ for every $i = 1, 2, \ldots, n$. Let $F = \bigoplus_{i=1}^{n} F_i$ and $f = \bigoplus_{i=1}^{n} f_i$. Then $f : F \longrightarrow R/J(R)$ is a flat e-cover of $R/J(R)$ by Lemma 2.2. Since R is projective, there is a homomorphism $g : R \longrightarrow F$ satisfying $fg = \pi$ where $\pi : R \longrightarrow R/J(R)$ is the canonical epimorphism. Since $\operatorname{Im} g + \operatorname{Ker} f = F$ and $\operatorname{Ker} f \ll_e F$, $F = \operatorname{Im} g \oplus S$ for some semisimple module S by [21, Proposition 2.3]. Then S is a flat semisimple module as a direct summand of F and hence it is projective by the assumption. We have that $\operatorname{Im} g \cong R/\operatorname{Ker} g$ is likewise flat and $\operatorname{Ker} g \leq \operatorname{Ker} \pi = J(R)$ which imply that $\operatorname{Ker} g = 0$ by [16, Exercise 4.20]. So $\operatorname{Im} g \cong R$ is projective. Then F_i is projective for every $i = 1, 2, \ldots, n$ for being a direct summand of the projective module $F = \operatorname{Im} g \oplus S$. Therefore every simple module has a projective e-cover. The rest of the proof follows from Proposition 2.13 and [9, Corollary 4.3]. □

It is known by [18, Fact 7.5] that over a semilocal commutative ring every simple flat module is projective. Using this fact and Theorem 3.6, we obtain the following result.

Corollary 3.7. *Let R be a semilocal commutative ring. Then R is semiperfect if and only if every simple R-module has a flat e-cover.*

In the case of commutative domains, it is easy to determine the rings whose simple modules have flat e-covers. Recall that a ring R is called *local* if $J(R)$ is a maximal ideal.

Proposition 3.8. *Let R be a commutative domain. Then R is local if and only if every simple R-module has a flat e-cover.*

Proof. ($\Rightarrow$) It is clear.

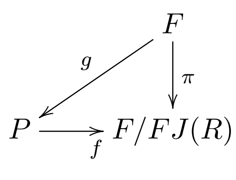
($\Leftarrow$) Let F be a flat e-cover of a simple module. By Corollary 2.6, we may assume that F is cyclic. Then $F \cong R$, since F is torsion-free and so there is a maximal ideal M of R with $M \ll_e R$. Therefore R is local. □

Observe from Proposition 3.8 that $\mathbb{Z}$ is not an e-perfect ring. The following theorem generalizes a result due to N. Ding and J. Chen (see [12, Theorem 4]).

Theorem 3.9. *A ring R is right perfect if and only if it is semilocal and every semisimple R-module has a flat e-cover.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Let F be a free module. Since $R/J(R)$ is semisimple, $F/FJ(R)$ is a semisimple right R-module. By assumption $F/FJ(R)$ has a flat e-cover $f : P \longrightarrow F/FJ(R)$ for some flat right R-module P. Since F is projective, we have the commutative diagram



where $\pi : F \longrightarrow F/FJ(R)$ is the canonical epimorphism. Since π is an epimorphism, we have $\operatorname{Ker} f + \operatorname{Im} g = P$. Since $\operatorname{Ker} f \ll_e P$, $\operatorname{Im} g$ is a direct summand of P by [21, Proposition 2.3] and so $\operatorname{Im} g$ is flat. Then $\overline{f} : \operatorname{Im} g \longrightarrow F/FJ(R)$ induced by f is a flat e-cover of $F/FJ(R)$, since $\operatorname{Ker} \overline{f} = \operatorname{Ker} f \cap \operatorname{Im} g \ll_e \operatorname{Im} g$ by Lemma 2.4. Since F is projective, $F/\operatorname{Ker} g \cong \operatorname{Im} g$ is flat and $\operatorname{Ker} g \leq \operatorname{Ker} \pi = FJ(R) = \operatorname{Rad} F$, we have $\operatorname{Ker} g = 0$ by [16, Exercise 4.20], so $\tilde{g} : F \longrightarrow \operatorname{Im} g$ induced by g is an isomorphism. $\operatorname{Rad} F = FJ(R) = \tilde{g}^{-1}(\operatorname{Ker} f \cap \operatorname{Im} g) \ll_e F$ by [21, Proposition 2.5]. Lemma 2.8 implies that $\operatorname{Rad} F \ll F$. By [3, Lemma 28.3], $J(R)$ is right T-nilpotent. Hence R is right perfect. □

Corollary 3.10. *A ring R is right perfect if and only if it is semilocal and right e-perfect.*

4. Flat-locally projective covers

As we have cited at the beginning of this study, a flat cover of a module M need not be a (locally) projective cover of the module M. Conversely, even if a module has a (locally) projective cover, that cover need not be a flat cover in general. We now give some examples for these cases.

Example 4.1.

(1) *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$, where $n > 1$. Then flat covers of M are not locally projective covers of M. Suppose that flat cover $\alpha : F \longrightarrow M$ of M is a locally projective cover of M. Since M is cyclic, it follows from* [20, Lemma 3.3] *that F is a cyclic projective $\mathbb{Z}$-module and so $F \cong \mathbb{Z}$. Since* $\operatorname{Rad}(\mathbb{Z}) = 0$, *we obtain that* $\operatorname{Ker} \alpha = 0$ *and $\mathbb{Z} \cong M$, a contradiction.*

(2) *Let p be a prime integer and consider the local ring $R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } p \nmid b \right\}$. Then the canonical epimorphism $\pi : R \longrightarrow R/J(R)$ is a (locally) projective cover of the right R-module $R/J(R)$ but not a flat cover of the module $R/J(R)$ by* [8, Proposition 2.13].

Definition 4.2. *Let M be a module. M is said to have a* flat-locally projective cover *(shortly, an FLP*-cover*) if a flat cover $\alpha : F \longrightarrow M$ of M is also a locally projective cover of M, and M is said to have a* flat-generalized locally projective cover *(shortly, an $FGLP$*-cover*) if a flat cover $\alpha : F \longrightarrow M$ of M is also a generalized locally projective cover of M. In these cases, we also say that F is an FLP-cover and an $FGLP$-cover of M, respectively.*

The concepts of FLP-cover and $FGLP$-cover coincide for the class of projective modules as [19, Theorem 1.2.12] indicates. Over a semilocal ring, we have the following result.

Lemma 4.3. *Let R be a semilocal ring and S be a simple R-module whose flat covers are locally projective. Then S has an $FGLP$-cover.*

Proof. Let $\alpha : F \longrightarrow S$ be a flat cover of the simple module S. Since R is semilocal, it follows from [15, Theorem 9.3.5] and [14, Corollary 23] that $\mathrm{Rad}(F) = FJ(R) = \mathrm{Ker}\,\alpha$. Then F is locally projective and $\mathrm{Ker}\,\alpha = \mathrm{Rad}(F)$ is a maximal submodule of F so that $\alpha : F \longrightarrow S$ is an $FGLP$-cover of M. □

It is well known that a ring R is semiperfect if and only if every simple R-module has a generalized locally projective cover. Now we have the next result.

Proposition 4.4. *The following statements are equivalent for a ring R.*

(1) *Every simple R-module has an $FGLP$-cover.*
(2) *R is semiperfect and flat covers of simple R-modules are locally projective.*
(3) *R is semilocal and flat covers of simple R-modules are locally projective.*

Proof. (1) ⇒ (2) By (1), every simple module has a generalized locally projective cover and so R is semiperfect. The rest of (2) is clear.

(2) ⇒ (3) follows from the fact that semilocal rings are proper generalizations of semiperfect rings.

(3) ⇒ (1) is a consequence of Lemma 4.3. □

Note that a flat cover of a finitely generated module need not be finitely generated in general. Xue proved in [20, Lemma 3.3] that if a finitely generated module M has a locally projective cover $\alpha : P \longrightarrow M$, then P is a finitely generated projective module. Using this fact, we have the following result showing that FLP-covers of a finitely generated module are finitely generated.

Corollary 4.5. *Let $\alpha : F \longrightarrow M$ be an FLP-cover of a module M. If M is finitely generated, then F is a finitely generated projective module.*

Proof. Since $\alpha : F \longrightarrow M$ is an FLP-cover, it is a locally projective cover of M. By [20, Lemma 3.3], we obtain that F is a finitely generated projective module. □

In [8], a ring R is called right B-perfect if for every flat R-module F, simple right R-module S and homomorphisms $f : R \longrightarrow S$, $h : F \longrightarrow S$, there exists a homomorphism $g : F \longrightarrow R$ such that $h = fg$. It follows from [8, Theorem 2.4] that a ring R is right B-perfect if and only if flat covers of simple R-modules are projective. Now we give a characterization of right B-perfect rings via FLP-covers.

Theorem 4.6. *A ring R is right B-perfect if and only if every simple R-module has an FLP-cover.*

Proof. (⇒) For any simple module S, let $\alpha : F \longrightarrow S$ be a flat cover of S. It follows from the assumption that F is projective. It means that $\alpha : F \longrightarrow M$ is an FLP-cover of S.

(⇐) It follows from Corollary 4.5 and [8, Theorem 2.4]. □

In [2], a ring R is called *right A-perfect* if every flat right R-module is R-projective. The following proper implications on rings hold.

$$\text{right perfect} \Longrightarrow \text{right } A\text{-perfect} \Longrightarrow \text{right } B\text{-perfect} \Longrightarrow \text{semiperfect} \Longrightarrow \text{semilocal}$$

It is shown in [2, Theorem 3.7] that a ring R is right A-perfect if and only if flat covers of finitely generated R-modules are projective (that is, they are projective covers). Now we generalize this fact as follows.

Theorem 4.7. *The following statements are equivalent for a ring R.*

(1) *R is right A-perfect.*
(2) *Every finitely generated R-module has an FLP-cover.*
(3) *Every cyclic R-module has an FLP-cover.*

Proof. (1) ⇒ (2) Flat covers of a finitely generated module are projective covers by assumption.

(2) ⇒ (3) is clear.

(3) ⇒ (1) is a consequence of Corollary 4.5 and [2, Theorem 3.7]. □

We now prove that rings whose modules have $FGLP$-covers are right perfect.

Theorem 4.8. *Let R be a ring. The following statements are equivalent.*

(1) *R is right perfect.*
(2) *Flat covers of R-modules are locally projective.*
(3) *Every R-module has an FLP-cover.*
(4) *Every R-module has an $FGLP$-cover.*

Proof. (1) ⇒ (3) Let M be a module and $\alpha : F \longrightarrow M$ be a flat cover of M. Since R is right perfect, this flat cover is also a projective cover and so a locally projective cover of M. Hence M has an FLP-cover as desired.

(3) ⇒ (4) It is clear by definition.

(4) $\Rightarrow$ (2) Let M be a module and $\alpha : F \longrightarrow M$ be a flat cover of M. Since M has an $FGLP$-cover by (4), this flat cover of M is also a generalized locally projective cover of M. So, F is a locally projective module.

(2) $\Rightarrow$ (1) Let F be a flat module. By (2), we have that F is locally projective. Hence R is right perfect by [23, p. 60]. □

Let R be a ring. If every semisimple R-module has a projective cover, then R is right perfect. We do not know if the condition "projective cover" can be weakened to "flat-locally projective cover" here. The next result is interesting.

Theorem 4.9. *The following statements are equivalent for a ring R.*

(1) *R is right perfect.*
(2) *Every semisimple R-module has an FLP-cover.*

Proof. (1) $\Rightarrow$ (2) is a consequence of Theorem 4.8.

(2) $\Rightarrow$ (1) Since every simple R-module is semisimple, it follows from Theorem 4.6 that R is right B-perfect and so it is semilocal. [11, Theorem 3.9] implies that R is right perfect. □

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