

# Large transformations with moderate strains of tensile membrane structures

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## Abstract

Using a classical non-linear theory, we analytically investigate possible ways for transforming the shape of a curved elastic membrane while keeping it tensioned and moderately strained. This is a critical issue because, as a rule, membranes must be considerably stretched in order to avoid wrinkling and slackening. If the final configuration is fixed, the membrane can be cut and formed according to the final shape, but this cannot be done if more configurations, considerably distant from one another, have to be achieved. Nevertheless, we propose large transformation movements that can be obtained starting from flat membranes while maintaining their strain as limited. We discuss in detail the paradigmatic example of the hyperbolic-paraboloid-shaped membrane. These opportunities are suitable for applications of transformable architecture because they do not require excessive tensioning, compatible with the strength of materials used for this kind of structures.

## Keywords

Transformable architecture, kinetic structures, tensile membrane, non-linear deformations, adaptable structures, analytical methods, optimization

## 1. Introduction

The exploration of kinetic architectural structures moves between the constraints of sustainability, on the one side, and the opportunities given by design and manufacturing, on the other side. Within this field of research, the most recent and compelling designs are *smart structures* that modify their working conditions with respect to changeable external loads [1], or *transformable structures* that change their shape to comply with different-in-type tasks.

This study investigates the design of transformable tensioned membranes, that is, tensile structures that can move from one equilibrium configuration to another, characterized by different states of stress and highly different geometric configuration. Tensile membrane structures are positively considered because of their lightness and their efficient use of the material, which makes them particularly suitable for kinetic transformations. Certainly, lightweight architecture has a very long tradition, which starts with tents [2] and passes through a wide range of temporary shelters for emergency and entertainment [3,4]. Most of these constructions are transportable, so that design and engineering are optimized for an easy dismissing and relocation. Nevertheless, with the increasing popularity of sport and music events,

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there has been a new demand for transformable facilities to make such outdoor events enjoyable throughout the whole year. In particular, a wide set of roofs for stadiums has been built, which are mainly composed of panels sliding on rails and, in general, covered with all-framed, slightly curved membranes [5,6]. In these cases, the membrane moves rigidly with its frame, but other mechanisms have been proposed for small roofs where the joints can slide along tensioned cables [7].

Noteworthy applications of the same kind are the umbrella-like mechanisms, in particular the very remarkable examples designed by Bodo Rash [8]. In general, most of the deployable skeletons used for medium-size roofs and shelters are scissor-like mechanisms with soft covers. Starting from the transportable/foldable theatre designed by Pinero [9], various designs have been proposed for swimming pools and other facilities, where the membrane is point supported at the upper (or lower) vertexes of the mechanism [10]. For small-size elements such as portable tents or (in a completely different field of application) space antennas [11], also foldable tensegrity structures have been analysed and constructed [12]. Other interesting mechanisms that can support membranes are the modular Bennett-like mechanism [13–15] and the cable-stiffened voussoirs [16]. Other studies have regarded optic [17] and space [18] applications, but for these the boundary and load conditions are generally different from those associated with architectural applications.

The aforementioned proposals mainly focus upon the supporting foldable mechanism, but little or no attention has been paid to the behaviour of the soft cover itself. In practice, the membrane is supposed to be either folded or tensioned once the (unique) working configuration has been reached. The major attention is paid to avoiding tearing of the membrane during the folding process, but more than one working shape is not considered [19,20].

In the present article, the shaping material is required to be compatible with various working conditions, geometrically distant from one another. In other words, we are not presenting a further application of soft covers to kinetic frames, but a truly transformable membrane structure. This is, in our opinion, an open and up-to-date research field. In order to increase the load-bearing capacity, an anticlastic doubly curved surface is usually preferred, where the state of stress is close to the ideal uniform and equi-biaxial condition (minimal surface). However, as will be demonstrated in the following, this configuration cannot be achieved with an initially plane sheet, because the strain demand associated with the pure geometric distortion would remain far from homogeneous. In order to limit this “spurious” behaviour, the standard technique consists of cutting the sheets and assembling them, so as to form a shape that, in its unstressed state, is very close to the working condition. If the *so-shaped* membrane is uniformly stretched into two orthogonal directions, the shape can be substantially maintained and the state of stress is, in practice, equi-biaxial.

However, if the membrane needs to be transformed, one reference configuration, curved and unstressed, cannot be selected because it would be dominant with respect to any other. Instead, we aim at achieving geometrically distant configurations consequent to different-in-type states of stress, compatible with the material strength, which are alternatively assigned at the boundary of the membrane so as to make it deform consequently. Each of these configurations is characterized by a dominant direction of curvature, and the ratio between the two main curvatures can be optimized in order to give to the membrane the desired stiffness, without inducing excessive strains. This allows using an initially flat membrane sheet, which is then simply transformed without cuts. The proposed histories of deformation can limit the strain demand required by the geometric distortions, and therefore allow for the use of materials with relatively high elastic moduli, which are preferred for this kind of structure.

Investigating the movement of a membrane under large deformations is a well-developed subject, but more complicated than the analysis of small deformations. Most solutions in closed form, such as the classical Hencky’s problem [21], correspond to initially flat and unstressed circular elastic membranes, clamped at the boundaries and subject to a uniform differential pressure. This problem was further developed in [22] by accounting for tangential components of the external loads. Other rectangular and axi-symmetrical configurations were investigated with approximate analytical methods and/or numerical approaches [23,24], and extended to very general cases [25].

In Section 2, we present the model used for the membrane under the hypotheses of small strains and large rotations, the same also adopted by other authors [26]. The state of stress must be tensile everywhere in order to avoid slackening or wrinkling [27,28], but attempts to modify the shape of a tensioned membrane face the tearing of the membrane or the generation of wrinkles, which in general cannot be

avoided unless the material is highly stretched. Although in some cases limited wrinkles may be acceptable and controllable [29], in general wrinkles and tears are both highly risky because they may limit the practical usability and they may create pounding under snow or rain. We present results of very general relevance for an initially flat membrane, showing in particular that the only deformations compatible with homogeneous states of no-compression stress are represented by surfaces with zero Gaussian curvature. Such states can be achieved, at least in principle, with very moderate straining of the membrane because they are associated with developable surfaces. For this special class of deformations, the direction and amount of curvature can be changed with no strain demand, so as to obtain various different-in-type surfaces.

These configurations are ideal, because the membrane would be tensionless in the direction of the non-zero principal curvature. This is why in Section 2, following the paradigmatic example of a transformable double-curvature hyperbolic paraboloid (hypar), we study how to approximate such limit conditions with a biaxially tensioned membrane. In particular, the two principal curvatures will have opposite signs and, in absolute value, one will be much greater than the other, so that the resulting surface will be “almost” developable. Correspondingly, the tensile stress in the direction of the minimal (in absolute value) curvature will be much higher than in the direction of the maximal curvature. This class of shapes demands limited strain for geometric deformations, and therefore does not need to be highly tensioned. However, without modifying the curved shape, an additional, properly calibrated, state of stress can be added to the basic one in order to increase the out-of-plane stiffness of the membrane, if required. In conclusion, we define and optimize the set of those configurations, and related geometrical constraints, in which the membrane can be alternatively transformed without excessive wrinkling and tearing.

Of course, moving from the theoretical investigation to the practical implementation poses basic problems. The kind of fabric, and especially its orientation, the way to solve the extra functions asked of the tensioning apparatus and the movement capability of the supporting structures are all aspects to be properly considered. Although the details of these requirements are outside the scope of the present article, a brief presentation of the main issues is given in the concluding section with reference to a physical model. Indeed, the new design permitted by the added usability and performances may, by itself, further enrich the aesthetical and reliability qualities of the whole system, whose possible developments are yet to be explored.

## 2. The model

Let  $\Omega \subset \mathbb{R}^2$  represent the undistorted planar reference configuration of an elastic membrane. Introduce the reference system  $(x,y,z)$ , such that  $\Omega$  is in the plane  $(x,y)$ , and let  $u(x,y)$ ,  $v(x,y)$  and  $w(x,y)$  represent the components of displacement of a particle of the membrane, initially at  $(x,y,0)$ , in the direction of the axes  $x$ ,  $y$  and  $z$ , respectively.

### 2.1 Variational approach and governing equations

The components of the Green–Lagrange strain associated with a distortion of the membrane take the form

$$\begin{aligned} e_{xx} &= \frac{1}{2} \left[ (1 + u_{,x}^2) + v_{,x}^2 + w_{,x}^2 - 1 \right], \\ e_{yy} &= \frac{1}{2} \left[ (1 + v_{,y}^2) + u_{,y}^2 + w_{,y}^2 - 1 \right], \\ e_{xy} &= \frac{1}{2} \left[ (1 + u_{,x})u_{,y} + (1 + v_{,y})v_{,x} + w_{,x}w_{,y} \right], \end{aligned} \tag{1}$$

where the comma denotes partial differentiation with respect to the variable indicated afterwards, for example,  $u_{,x} \equiv \frac{\partial u}{\partial x}$ . If  $\varepsilon$  represents an order parameter of smallness, let us suppose that  $w_{,x} = O(\varepsilon)$  and  $w_{,y} = O(\varepsilon)$  and that  $u_{,x} = o(\varepsilon)$ ,  $u_{,y} = o(\varepsilon)$ ,  $v_{,x} = o(\varepsilon)$ ,  $v_{,y} = o(\varepsilon)$ . Thus if one neglects, according to the classical von Kármán approach, terms of order higher than  $\varepsilon$ , then (1) becomes

$$\begin{aligned}
e_{xx} &= u_{,x} + \frac{1}{2}w_{,x}^2, \\
e_{yy} &= u_{,y} + \frac{1}{2}w_{,y}^2, \\
e_{xy} &= \frac{1}{2}[u_{,y} + v_{,x} + w_{,x}w_{,y}].
\end{aligned} \tag{2}$$

The membrane is supposed to be linear elastic, homogeneous and isotropic, so that the components of stress take the form

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2} (e_{xx} + \nu e_{yy}), \\
\sigma_{yy} &= \frac{E}{1-\nu^2} (e_{yy} + \nu e_{xx}), \\
\tau_{xy} &= 2G e_{xy},
\end{aligned} \tag{3}$$

where  $E$  is the Young's modulus,  $\nu$  is the Poisson's ratio and  $G = E/[2(1 + \nu)]$  is the shear modulus.

Although not used here, we observe in passing that this model comprehends, with minor modifications, the response of orthotropic membranes in which the directions of elastic symmetry are parallel to the coordinate axes. Therefore, the forthcoming deductions would be compatible with orthogonally patterned fabrics, that is, fabrics reinforced by two families of fibres parallel to the  $x$  and  $y$  axes [30]. Of course, in the case of an orthotropic material, one would have that  $G \neq E/[2(1 + \nu)]$  in general.

The elastic strain energy of the membrane is of the form

$$\Theta[u, v, w] = \frac{E s}{2(1-\nu^2)} \int_{\Omega} [e_{xx}^2 + e_{yy}^2 + 2\nu e_{xx}e_{yy} + 2(1-\nu)e_{xy}^2] dA, \tag{4}$$

where  $s$  denotes the (small) thickness of the membrane, supposed to be constant, and the relationship between the displacement components  $u$ ,  $v$  and  $w$  and the strain components  $e_{xx}$ ,  $e_{yy}$ ,  $e_{xy}$  is given by (2). We suppose that, in the reference state  $\Omega$ , traction forces per unit area of components  $t_x$ ,  $t_y$  and  $t_z$  are applied on the part  $\partial_t\Omega$  of the membrane border  $\partial\Omega$ , whereas on the complementary part  $\partial_u\Omega = \partial\Omega \setminus \partial_t\Omega$  the displacement components

$$u = \bar{u}, \quad v = \bar{v}, \quad w = \bar{w}, \quad \forall (x, y) \in \partial_u\Omega \tag{5}$$

are prescribed. The total energy functional thus reads

$$\Pi[u, v, w] = \frac{E s}{2(1-\nu^2)} \int_{\Omega} [e_{xx}^2 + e_{yy}^2 + 2\nu e_{xx}e_{yy} + 2(1-\nu)e_{xy}^2] dA - s \int_{\partial_t\Omega} (t_x u + t_y v + t_z w) dl. \tag{6}$$

Taking the first variation of this functional with respect to the variations  $u + \delta u$  and  $v + \delta v$ , one obtains the field equations

$$\begin{aligned}
\frac{Es}{1-\nu^2} (e_{xx} + \nu e_{yy})_{,x} + \frac{Es}{1+\nu} e_{xy,y} &= \sigma_{xx,x} + \tau_{xy,y} = 0, \quad \forall (x, y) \in \Omega, \\
\frac{Es}{1+\nu} e_{xy,x} + \frac{Es}{1-\nu^2} (e_{yy} + \nu e_{xx})_{,y} &= \tau_{xy,x} + \sigma_{yy,y} = 0, \quad \forall (x, y) \in \Omega,
\end{aligned} \tag{7}$$

together with the natural boundary conditions

$$\begin{aligned}
\frac{Es}{1-\nu^2} (e_{xx} + \nu e_{yy}) n_x + \frac{Es}{1+\nu} e_{xy} n_y &= \sigma_{xx} n_x + \tau_{xy} n_y = t_x, \quad \forall (x, y) \in \partial_t \Omega, \\
\frac{Es}{1+\nu} e_{xy} n_x + \frac{Es}{1-\nu^2} (e_{yy} + \nu e_{xx}) n_y &= \tau_{xy} n_x + \sigma_{yy} n_y = t_y, \quad \forall (x, y) \in \partial_t \Omega,
\end{aligned} \tag{8}$$

where  $n_x$  and  $n_y$  denote the components of the outer unit normal to  $\partial\Omega$  in the  $(x,y)$  plane. Taking the variation  $w + \delta w$ , the first variation of (6) gives

$$\Pi[u, v, w|w] = \frac{E s}{2(1 - \nu^2)} \int_{\Omega} [\sigma_{xx} w_{,x} \delta w_{,x} + \sigma_{yy} w_{,y} \delta w_{,y} + \tau_{xy} (w_{,x} \delta w_{,y} + w_{,y} \delta w_{,x})] dA - s \int_{\partial\Omega} t_z \delta w dl = 0, \tag{9}$$

which, considering (7), implies

$$\sigma_{xx} w_{,xx} + 2\tau_{xy} w_{,xy} + \sigma_{yy} w_{,yy} = 0, \quad \forall (x, y) \in \Omega, \tag{10}$$

and the boundary condition

$$(\sigma_{xx} w_{,x} + \tau_{xy} w_{,y}) n_x + (\tau_{xy} w_{,x} + \sigma_{yy} w_{,y}) n_y = t_z, \quad \forall (x, y) \in \partial_t \Omega. \tag{11}$$

In conclusion, the boundary value problem is completely defined by the equilibrium equations (7) and (10), the compatibility equations (2), the constitutive equations (3) and the boundary conditions (5), (8) and (11).

In particular, (10) represents the equilibrium in the  $z$ -direction of an elementary portion of the membrane. It is then clear that in this theory, where the membrane strains are given by (2), the curvatures are approximated by the second derivatives of the out-of-plane component of the displacement.

### 2.2 General results

General results, which will be useful for the forthcoming considerations, can be obtained from the simple model just presented. These can be summarized in the following three propositions.

**Proposition 1.** The only admissible deformations with non-negative Gaussian curvature that are compatible with a full tensile state of stress of the membrane are those with null Gaussian curvature. The stress tensor and the curvature tensor are coaxial, and the tensile stress is null in the direction of non-null principle curvature.

In order to prove this statement recall that, in the model of Section, the curvatures are approximated with the second derivatives of the out-of-plane displacement  $w(x,y)$ , so that the components of the curvature tensor  $1/R_x, 1/R_y, 1/R_{xy}$  take the form

$$\frac{1}{R_x} = w_{,xx}, \quad \frac{1}{R_y} = w_{,yy}, \quad \frac{1}{R_{xy}} = w_{,xy}. \tag{12}$$

At each point  $(x,y)$  it is possible to define the principal directions of curvature, identified by the unit vectors  $\mathbf{r}_1(x,y)$  and  $\mathbf{r}_2(x,y)$ ,  $\mathbf{r}_1(x,y) \cdot \mathbf{r}_2(x,y) = 0$ . If  $\xi$  and  $\eta$  denote the coordinate in a local reference frame  $(\xi,\eta)$  whose axes are parallel to  $\mathbf{r}_1(x,y)$  and  $\mathbf{r}_2(x,y)$  at each point  $(x,y)$ , the principal curvatures  $1/R_\xi, 1/R_\eta$  read

$$\frac{1}{R_\xi} = w_{,\xi\xi}, \quad \frac{1}{R_\eta} = w_{,\eta\eta}, \quad \text{with} \quad \frac{1}{R_{\xi\eta}} = w_{,\xi\eta} = 0. \tag{13}$$

Now let  $\mathbf{t}_1(x,y)$  and  $\mathbf{t}_2(x,y)$ ,  $\mathbf{t}_1(x,y) \cdot \mathbf{t}_2(x,y) = 0$ , denote the unit vectors that indicate, at each point  $(x,y)$ , the principal directions of stress, with associated principal stress components  $\sigma_1(x,y)$  and  $\sigma_2(x,y)$ . Since in general  $\mathbf{t}_1(x,y) \neq \mathbf{r}_1(x,y)$ , let  $\alpha(x,y)$  represent the angle formed by  $\mathbf{t}_1(x,y)$  and  $\mathbf{r}_1(x,y)$  or, equivalently, by  $\mathbf{t}_2(x,y)$  and  $\mathbf{r}_2(x,y)$ .

Now, expression (10) implies that the scalar product between the stress tensor and the curvature tensor is null. In terms of principal components of stress and curvature, this equation takes the form

$$\begin{aligned} & \frac{\sigma_1}{R_\xi} \cos^2 \alpha + \frac{\sigma_1}{R_\eta} \sin^2 \alpha + \frac{\sigma_2}{R_\xi} \sin^2 \alpha + \frac{\sigma_2}{R_\eta} \cos^2 \alpha \\ & = \left( \frac{\sigma_1}{R_\xi} + \frac{\sigma_2}{R_\eta} \right) \cos^2 \alpha + \left( \frac{\sigma_1}{R_\eta} + \frac{\sigma_2}{R_\xi} \right) \sin^2 \alpha = 0, \quad \forall (x, y) \in \Omega. \end{aligned} \quad (14)$$

Now, by hypothesis,  $\sigma_1 \geq 0$ ,  $\sigma_2 \geq 0$  and  $1/R_\xi, 1/R_\eta \geq 0$ . Let us suppose, without losing generality, that  $\sigma_1 \geq \sigma_2 \geq 0$  and that  $1/R_\eta \geq 1/R_\xi \geq 0$  (the case in which both curvatures are not positive can be treated analogously). If  $\sigma_1/R_\eta = 0$ , then either both curvatures are null, or the stress is null. Both cases are trivial. Let us thus suppose that  $\sigma_1/R_\eta > 0$ . Then from (14),  $\sin \alpha = 0$ , that is,  $\mathbf{t}_1(x, y) = \mathbf{r}_1(x, y)$  and  $\mathbf{t}_2(x, y) = \mathbf{r}_2(x, y)$  for all  $(x, y)$ . Therefore, the stress tensor and the curvature tensor are coaxial. From (14) one further has  $\sigma_1/R_\xi + \sigma_2/R_\eta = 0$ , but since  $\sigma_1/R_\eta > 0$  and consequently  $\sigma_1 > 0$  and  $1/R_\eta > 0$ , one finds that  $\sigma_2 = 0$  and  $1/R_\xi = 0$ . This concludes the proof of the proposition.

It may be interesting to observe that the alternative form (14) of the equilibrium equation (10) can also be written as

$$\left( \sigma_1 \cos^2 \alpha + \sigma_2 \sin^2 \alpha \right) \frac{1}{R_\xi} + \left( \sigma_1 \sin^2 \alpha + \sigma_2 \cos^2 \alpha \right) \frac{1}{R_\eta} = \frac{\sigma_\xi}{R_\xi} + \frac{\sigma_\eta}{R_\eta} = 0, \quad \forall (x, y) \in \Omega, \quad (15)$$

where  $\sigma_\xi$  and  $\sigma_\eta$  represent the normal component of stress in the directions of the unit vectors  $\mathbf{r}_1(x, y)$  and  $\mathbf{r}_2(x, y)$ , that is, in the direction of the principal curvatures. In general this means that if the membrane is uniformly taut, so that  $\sigma_\xi = \sigma_\eta$ , then the mean curvature must be zero.

The following two propositions are very general and regard the compatibility of homogenous states of stress for the deformed membrane.

**Proposition 2.** Within the hypotheses of the model of Section 2.1, the only equilibrium states associated with a uniformly equi-biaxial state of stress are those for which  $w(x, y)$  is a linear function. In other words, the membrane cannot be deformed and achieve a uniformly equi-biaxial state of stress unless the deformed shape is trivially planar.

In order to prove the proposition, let us suppose that  $\sigma_{xx} = \sigma_{yy} = \sigma_0$  and  $\tau_{xy} = 0$ . Then, from (10) one finds that  $w(x, y)$  is harmonic since  $w_{,xx} + w_{,yy} = 0$ . Furthermore, from (2) and (3) one has

$$\begin{aligned} u_{,x} + \frac{1}{2} w_{,x}^2 &= \frac{1-\nu}{E} \sigma_0, \\ v_{,y} + \frac{1}{2} w_{,y}^2 &= \frac{1-\nu}{E} \sigma_0, \\ u_{,y} + v_{,x} + w_{,x} w_{,y} &= 0. \end{aligned} \quad (16)$$

We can thus demonstrate that also  $u(x, y)$  and  $v(x, y)$  are harmonic. In fact, differentiating the first equation and the second equation with respect to  $x$ , one has, respectively,  $u_{,xx} = -w_{,x} w_{,xx}$  and  $v_{,xy} = -w_{,y} w_{,xy}$ . Then, differentiating the third equation with respect to  $y$ , considering that  $w_{,xx} = -w_{,yy}$ , and substituting, one obtains

$$0 = u_{,yy} + v_{,xy} + w_{,xy} w_{,y} + w_{,x} w_{,yy} = u_{,yy} - w_{,xy} w_{,y} + w_{,xy} w_{,y} + w_{,x} w_{,yy} = u_{,yy} - w_{,x} w_{,xx} = u_{,yy} + u_{,xx}. \quad (17)$$

Analogously, differentiating the first equation and the second equation of (16) with respect to  $y$ , one finds  $u_{,xy} = -w_{,x} w_{,xy}$  and  $v_{,yy} = -w_{,y} w_{,yy}$ . Then, differentiating the third equation with respect to  $x$ , and substituting, one obtains

$$0 = u_{,xy} + v_{,xx} + w_{,xx} w_{,y} + w_{,x} w_{,xy} = -w_{,xy} w_{,x} + v_{,xx} + w_{,xx} w_{,y} + w_{,x} w_{,xy} = v_{,xx} - w_{,y} w_{,yy} = v_{,xx} + v_{,yy}. \quad (18)$$

Consider then the identity

$$(u,x),yy + (v,y),xx = (u,y + v,x),xy. \tag{19}$$

Then, from (16), one has

$$0 = (-\frac{1}{2}w_{,x}^2),yy + (-\frac{1}{2}w_{,y}^2),xx + (w_{,x} + w_{,y}),xy = (w_{,xy})^2 - w_{,xx}w_{,yy}. \tag{20}$$

However, since  $(w_{,xy})^2 \geq 0$  and  $w_{,xx} = -w_{,yy}$  one has that

$$-(w_{,xx})^2 \geq 0 \Rightarrow w_{,xx} = 0, \quad -(w_{,yy})^2 \geq 0 \Rightarrow w_{,yy} = 0. \tag{21}$$

Hence, recalling from (16) that  $u_{,xx} = -w_{,x} w_{,xx}$  and  $v_{,yy} = -w_{,y} w_{,yy}$ , and that both  $u(x,y)$  and  $v(x,y)$  are harmonic, one finally finds the conditions

$$u_{,xx} = v_{,xx} = w_{,xx} = u_{,yy} = v_{,yy} = w_{,yy} = 0. \tag{22}$$

It can be easily shown that if  $f(x,y): \Omega \rightarrow \mathbb{R}$  is a generic function such that  $f_{,xx} = f_{,yy} = 0$ , then  $f(x,y)$  is of the form  $f(x,y) = k_1xy + k_2x + k_3y + k_4$ ,  $k_1, k_2, k_3$  and  $k_4$  being constants.<sup>1</sup> Then one obtains

$$\begin{aligned} u(x,y) &= k_{u1}xy + k_{u2}x + k_{u3}y + k_{u4}, \\ v(x,y) &= k_{v1}xy + k_{v2}x + k_{v3}y + k_{v4}, \\ w(x,y) &= k_{w1}xy + k_{w2}x + k_{w3}y + k_{w4}. \end{aligned} \tag{23}$$

Finally, substituting these expressions into the third expression of (16), one has the condition

$$k_{u1}x + k_{u3} + k_{v1}y + k_{v2} + (k_{w1}y + k_{w2})(k_{w1}x + k_{w3}) = 0. \tag{24}$$

Differentiating this equation with respect to  $x$  and to  $y$ , one readily obtains  $k_{w1} = 0$ , which once substituted into (24) gives  $k_{u1}x + k_{v1}y + k_{u3} + k_{v2} + k_{w2}k_{w3} = 0$ . Differentiating this equation with respect to  $x$  and with respect to  $y$ , the result is that  $k_{u1} = k_{v1} = 0$ . In conclusion, one finally finds that

$$k_{u1} = k_{v1} = k_{w1} = 0, \tag{25}$$

which proves that the deformation associated with (24) is a trivial transformation of the membrane into a uniformly stretched plane. This concludes the proof of Proposition 2.

The third proposition, which defines the compatibility of more generic homogeneous states of stress for the deformed membrane, is at the base of the considerations of the next section.

**Proposition 3.** The only admissible deformations of the membrane that are compatible with a homogenous tensile state of stress are those with null Gaussian curvature. Again, the stress tensor and the curvature tensor are coaxial, and the tensile principal stress is null in the direction of non-null principle curvature.

To prove the third proposition, let us suppose, without losing generality, that the principal directions of tensile stress coincide with the directions of the axes  $x$  and  $y$ , that is,  $\sigma_{xx} = \sigma_1$ ,  $\sigma_{yy} = \sigma_2$  and  $\tau_{xy} = 0$ . Thus, the counterpart of (16) for this case becomes

$$\begin{aligned} u_{,x} + \frac{1}{2}w_{,x}^2 &= \frac{\sigma_1 - \nu\sigma_2}{E}, \\ v_{,y} + \frac{1}{2}w_{,y}^2 &= \frac{\sigma_2 - \nu\sigma_1}{E}, \\ u_{,y} + v_{,x} + w_{,x}w_{,y} &= 0. \end{aligned} \quad (26)$$

Differentiating these equations with respect to  $x$  and to  $y$ , one finds again  $u_{,xx} = -w_{,x}w_{,xx}$ ,  $v_{,xy} = -w_{,y}w_{,xy}$ ,  $u_{,xy} = -w_{,x}w_{,xy}$  and  $v_{,yy} = -w_{,y}w_{,yy}$ . Considering the equality (19) and using these conditions, one obtains analogously to (20) that

$$w_{,xx}w_{,yy} = (w_{,xy})^2. \quad (27)$$

From (10), one finally concludes that  $w(x,y)$  is such that

$$\sigma_1 w_{,xx} + \sigma_2 w_{,yy} = 0. \quad (28)$$

If both  $\sigma_1 > 0$  and  $\sigma_2 > 0$ , one has that  $w_{,yy} = -(\sigma_1/\sigma_2)w_{,xx}$ . With a similar argument that had given (21), one finds again that  $w_{,xx} = w_{,yy} = 0$ . Reasoning as in Proposition 1, the result is that the only admissible deformation is the trivially planar transformation, with associated zero Gaussian curvature.

Without losing generality, let us suppose then that  $\sigma_1 = 0$  and  $\sigma_2 > 0$ . From (28) one finds that  $w_{,yy} = 0$ . Then, from (27),  $w_{,xy} = 0$ . This proves that the stress tensor and the curvature tensor are coaxial. Moreover, the tensile principal stress is null in the direction of non-null principle curvature.

Furthermore, one finds that  $u_{,xx} = -w_{,x}w_{,xx}$ ,  $v_{,xy} = 0$ ,  $u_{,xy} = 0$  and  $v_{,yy} = 0$ . This implies that

$$v_{,y} = \text{const.}, \quad u_{,x} + \frac{1}{2}w_{,x}^2 = \frac{-\nu\sigma_2}{E} = \text{const.}, \quad u_{,x} = f(x), \quad w_{,x}^2 = \frac{-2\nu\sigma_2}{E} - f(x), \quad (29)$$

where  $f(x)$  is an arbitrary function of  $x$ . This concludes the proof of proposition 3.

### 3. Transformation movements for the hypar configuration

As shown in Section 2, homogeneous tensile states of stress in the initially flat membrane are compatible only with deformations characterized by null Gaussian curvature, being the stress and the curvature tensors coaxial. In particular, one of the principal stresses, that is, the one with the same direction of the non-null curvature, must be zero. This state represents an ideal condition for our purposes, because the membrane can achieve very large curvatures in one direction, with small tensile stresses in the other direction. In fact, if one neglects the loads orthogonal to the membrane, even a very small tensile stress can be applied in the direction of the null principal curvature, and this is sufficient to avoid slackening of the membrane. Of course, this ideal condition cannot be achieved in a real case, because the out-of-plane stiffness would be very low.

However, if one is interested in a transformable architecture, derived by alternative deformations sufficiently distant from one another, compatible with moderate straining of the membrane, the aforementioned results can represent a guide to select deformations that approximate the aforementioned ideal condition. This hypothesis is illustrated with reference to the paradigmatic example of a hyperbolic paraboloid (hypar).

Let the membrane be represented in its undistorted reference configuration by the square domain  $\Omega$  of equation  $|x| + |y| \leq a$ , and consider deformations of the form

$$w = \frac{A}{2a^2} [\alpha x^2 - (1 - \alpha)y^2 + (1 - \alpha)a^2], \quad 0 \leq \alpha \leq 1, \quad (30)$$

where  $A$  is a parameter having the dimension of a length, whereas  $\alpha$  is a number between 0 and 1. This is a hyperparabola, whose extreme points are  $w = A/2$  at  $(x,y) = (\pm a,0)$  and  $w = 0$  at  $(x,y) = (0, \pm a)$ . Within the approximation (12), one has that the directions of principal curvature are parallel to the axes  $x$  and  $y$ , and read

$$\frac{1}{R_x} = \frac{\alpha A}{a^2}, \quad \frac{1}{R_y} = -\frac{(1-\alpha)A}{a^2}, \quad \frac{1}{R_{xy}} = 0, \quad 0 \leq \alpha \leq 1. \tag{31}$$

Clearly, within the approximation of the model, the Gaussian curvature is null when either  $\alpha = 0$  or  $\alpha = 1$ . Our strategy here is to consider values of  $\alpha$  that are close to 0 or to 1, say  $\alpha = \alpha_0$  or  $\alpha = 1 - \alpha_0$ , with  $\alpha_0 \ll 1$ . This gives rise to two different-in-type configurations that are associated with anti-symmetrical states of strain in the membrane. Remarkably, the state with  $\alpha = \alpha_0$  can be obtained by tensioning the membrane much more in the  $x$ -direction than in the  $y$ -direction, and vice versa when  $\alpha = 1 - \alpha_0$ . Therefore, by simply inverting the state of stress, one can achieve two limit configurations, substantially far one-another and anti-symmetric, which are associated with moderate strains in the membrane, with no risk of slackening. Furthermore, one can obtain other compatible configurations if also the parameter  $A$  is allowed to vary together with  $\alpha$ .

### 3.1 Analysis of the hyper-like deformations

Substituting for  $u_{,x}$ ,  $v_{,y}$  and  $u_{,y} + v_{,x}$  from (2) into (19) one obtains

$$(\varepsilon_{xx} - \frac{1}{2}w_{,x}^2)_{,yy} + (\varepsilon_{yy} - \frac{1}{2}w_{,y}^2)_{,xx} = (2\varepsilon_{xy} - w_{,x}w_{,y})_{,xy}. \tag{32}$$

Therefore, inserting in this condition the expression (30), one finds

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy} + \frac{A^2}{a^4}\alpha(1-\alpha). \tag{33}$$

Using the constitutive equations (3), the last identity reads

$$(\sigma_{xx} - \nu\sigma_{yy})_{,yy} + (\sigma_{yy} - \nu\sigma_{xx})_{,xx} = 2(1+\nu)\tau_{xy,xy} + \frac{EA^2}{a^4}\alpha(1-\alpha). \tag{34}$$

It is convenient to use the Airy's stress function representation in the form

$$\sigma_{xx} = \phi_{,yy}, \quad \sigma_{yy} = \phi_{,xx}, \quad \tau_{xy} = -\phi_{,xy}, \tag{35}$$

so that (34) becomes

$$\phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = \frac{EA^2}{a^4}\alpha(1-\alpha). \tag{36}$$

Furthermore, equation (10), taking into account (30) and (35), reads

$$\alpha \phi_{,yy} - (1-\alpha)\phi_{,xx} = 0. \tag{37}$$

One can use this equation to simplify (36). In fact, differentiating (35) two times with respect to  $x$  and two times with respect to  $y$ , one has

$$\alpha \phi_{,yyyy} = (1-\alpha)\phi_{,xxyy}, \quad \alpha \phi_{,xxyy} = (1-\alpha)\phi_{,xxxx}, \tag{38}$$

and substituting into (36) for  $\phi_{,xxxx}$  and  $\phi_{,yyyy}$ , one finds

$$\phi_{,xxyy} = \frac{EA^2}{a^4}\alpha^2(1-\alpha)^2. \tag{39}$$

Then, still using (38) and substituting into (36), the resulting equations read

$$\phi_{,xxxx} = \frac{EA^2}{a^4} \alpha^3 (1 - \alpha), \quad (40)$$

and

$$\phi_{,yyyy} = \frac{EA^2}{a^4} \alpha (1 - \alpha)^3. \quad (41)$$

Through (35), equations (39)–(41) completely define the in-plane stress in the membrane.

The most general form of the Airy's function that verifies such equations<sup>2</sup> is a fourth-order polynomial, that is,

$$\begin{aligned} \phi(x, y) = & C_1 \frac{x^4}{24} + C_2 \frac{y^4}{24} + C_3 \frac{x^2 y^2}{4} \\ & + C_4 \frac{xy^3}{3} + C_5 \frac{x^3 y}{3} + C_7 \frac{x^3}{6} + C_8 \frac{y^3}{6} + C_9 \frac{xy^2}{2} + C_{10} \frac{x^2 y}{2} + C_{11} \frac{x^2}{2} + C_{12} \frac{y^2}{2} + C_{13} xy, \end{aligned} \quad (42)$$

where we have neglected, because it is inessential, the linear terms. Substituting into (39)–(41), one has that

$$C_1 = \frac{EA^2}{a^4} \alpha^3 (1 - \alpha), \quad C_2 = \frac{EA^2}{a^4} \alpha (1 - \alpha)^3, \quad C_3 = \frac{EA^2}{a^4} \alpha^2 (1 - \alpha)^2. \quad (43)$$

On the other hand, the equality (37) gives

$$\alpha \left[ C_2 \frac{y^2}{2} + C_3 \frac{x^2}{2} + 2C_4 xy + C_8 y + C_9 x + C_{12} \right] = (1 - \alpha) \left[ C_1 \frac{x^2}{2} + C_3 \frac{y^2}{2} + 2C_5 xy + C_7 x + C_{10} y + C_{11} \right], \quad (44)$$

which provides the further conditions

$$\begin{aligned} C_1 = \frac{EA^2}{a^4} \alpha^3 (1 - \alpha), \quad C_2 = \frac{EA^2}{a^4} \alpha (1 - \alpha)^3, \quad C_3 = \frac{EA^2}{a^4} \alpha^2 (1 - \alpha)^2, \\ \alpha C_4 = (1 - \alpha) C_5, \quad \alpha C_8 = (1 - \alpha) C_{10}, \quad \alpha C_9 = (1 - \alpha) C_7, \quad \alpha C_{12} = (1 - \alpha) C_{11}. \end{aligned} \quad (45)$$

In conclusion, the most general state of stress, compatible with the out-of-plane deformation (30) of the membrane, takes the form

$$\begin{aligned} \sigma_{xx} &= \frac{EA^2}{2a^4} \alpha (1 - \alpha)^2 [\alpha x^2 + (1 - \alpha)y^2] + (1 - \alpha)[2B_1 xy + B_2 y + B_3 x + B_4], \\ \sigma_{yy} &= \frac{EA^2}{2a^4} \alpha^2 (1 - \alpha) [\alpha x^2 + (1 - \alpha)y^2] + \alpha[2B_1 xy + B_2 y + B_3 x + B_4], \\ \tau_{xy} &= -\frac{EA^2}{2a^4} \alpha^2 (1 - \alpha)^2 xy - B_1 [\alpha x^2 + (1 - \alpha)y^2] - (1 - \alpha)B_3 y - \alpha B_2 x - B_5, \end{aligned} \quad (46)$$

where  $B_1, B_2, B_3, B_4, B_5$  are constants that are determined from the boundary conditions according to (8) and (11).

Remarkably, the state of stress can be divided into two parts. The first part is independent from the constants  $B_1 - B_5$  and it corresponds to the state of stress that is associated with the geometric demand due to the out-of-plane displacement (30). On the other hand, the second contribution represents a state of stress that can be “freely” added to the membrane without affecting its  $z$ -displacement. Only the aforementioned first part of the stress depends upon the elastic modulus of the membrane, and since

membranes are usually quite stiff, the only way to limit the stress from this quota is to choose appropriate target configurations.

For what concerns the second part, this is independent of the elasticity of the membrane and therefore represents a contribution that can be appropriately added for design purposes. For example, it may be calibrated, if necessary, to increase the stiffness of the deformed membrane itself and/or to compensate for possible wrinkles or inactive regions caused by the applied loads.

In particular, the state of stress represented by (46) should be associated with tensile stresses only and, to do so, it is necessary to set  $B_1 = B_2 = B_3 = B_5 = 0$  and  $B_4 \geq 0$ . The only surviving term, associated with  $B_4 > 0$ , represents a homogeneous state of stress in the membrane. In conclusion, the total stress is the sum of that part that is strictly necessary to fulfil the geometric demand, plus an arbitrary homogeneous tensile stress, whose principal directions coincide with the principal direction of curvature of the hyper and whose principal stress needs to be inversely proportional to the corresponding curvatures.

### 3.2 Changing curvatures while maintaining the fixed height of the corners

Suppose first that in (30) the parameter  $A$  is fixed, so that the four corner points  $(x,y) = (\pm a, 0)$  and  $(x,y) = (0, \pm a)$  do not change their height during the transformation. For the specific case  $A/2 = 2$  m and  $a = 5$  m, the two alternating configurations corresponding to  $\alpha = 0.2$  and  $\alpha = 0.8$  in (30) are plotted in Figure 1.

Let us consider the part of the stress in the membrane associated with the geometric demand and, therefore, set  $B_1 = B_2 = B_3 = B_4 = B_5 = 0$  in (46). We observe, first of all, that the quantity  $\alpha x^2 + (1-\alpha)y^2$  takes its maximum at two opposite corners of the membrane, whereas  $xy$  is maximum, in absolute value, on the extreme points on the lines  $x = \pm y$  in  $\Omega$ . Therefore, from (46), the maximum absolute value of the stress components  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\tau_{xy}$ , attained at different points, are respectively proportional to the quantities  $\alpha(1-\alpha)^3$ ,  $\alpha^2(1-\alpha)^2$ ,  $\alpha^2(1-\alpha)^2/2$ , when  $0 \leq \alpha \leq 1/2$ , and  $\alpha^2(1-\alpha)^2$ ,  $\alpha^3(1-\alpha)$ ,  $\alpha^2(1-\alpha)^2/2$ , when  $1/2 \leq \alpha \leq 1$ . The graphs of the corresponding functions are recorded in Figure 2. Here, one can appreciate that the shear stress  $\tau_{xy}$  remains in any case quite limited, attaining its maximum in the symmetric condition  $\alpha = 1/2$ , whereas the most severe configuration for the stress component  $\sigma_{xx}$  ( $\sigma_{yy}$ ) is when  $\alpha = 0.25$  ( $\alpha = 0.75$ ). The state of stress is of course minimal when either  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 1$ .

The maximal and minimal principal component of stress read

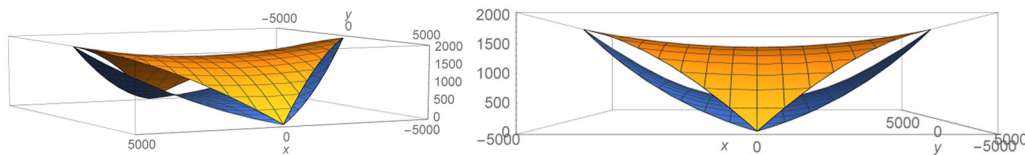


Figure 1. Deformed shape of the membrane (two views) for the two cases  $\alpha = 0.2$  and  $\alpha = 0.8$  ( $A/2 = 2$  m,  $a = 5$  m).

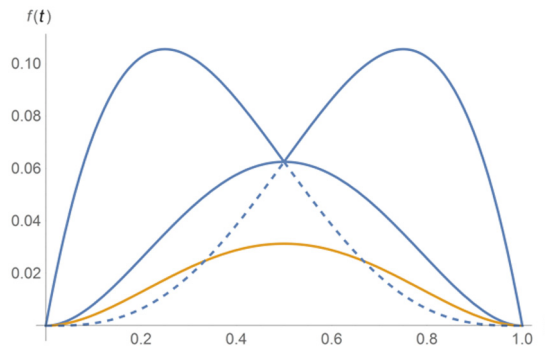
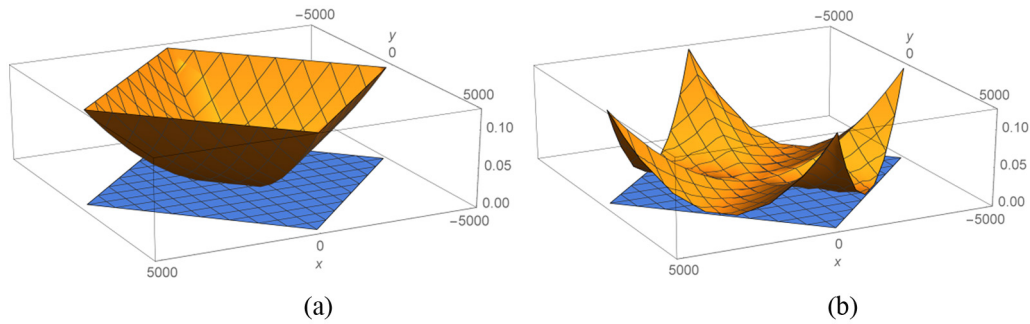
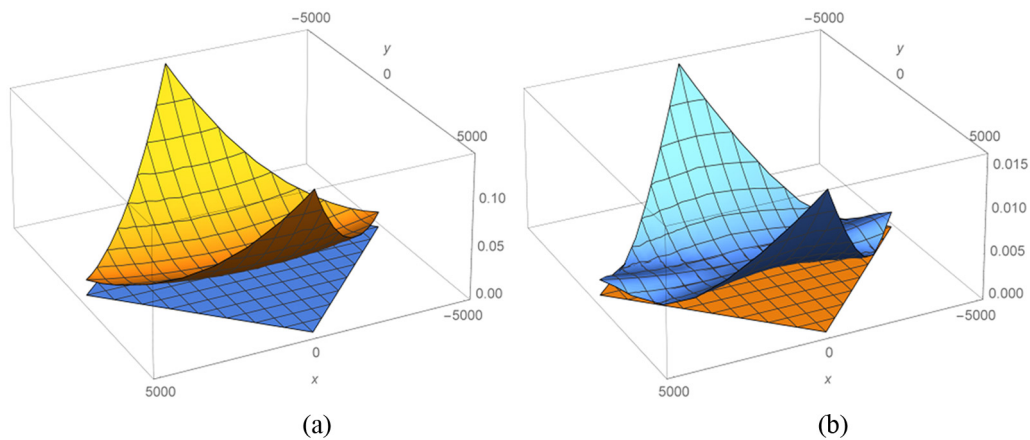


Figure 2. Plot of the functions  $\alpha(1-\alpha)^3$ ,  $\alpha^2(1-\alpha)^2$ ,  $\alpha^2(1-\alpha)^2/2$ ,  $\alpha^3(1-\alpha)$ ,  $0 \leq \alpha \leq 1$ , representative of the maximum absolute value of the stress components  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\tau_{xy}$ .



**Figure 3.** Normalized stresses (a)  $\sigma_{\max} 4a^2/(E A^2)$  and (b)  $\sigma_{\min} 4a^2/(E A^2)$  for the case  $\alpha = 1/2$ .



**Figure 4.** Normalized stresses (a)  $\sigma_{\max} 4a^2/(E A^2)$  and (b)  $\sigma_{\min} 4a^2/(E A^2)$  for the case  $\alpha = 0.1$ .

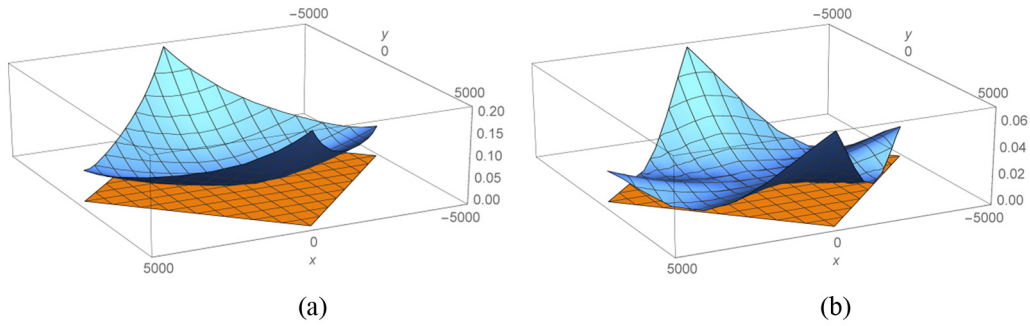
$$\sigma_{\max}, \sigma_{\min} = \frac{EA^2}{4a^4} \alpha(1 - \alpha) \left[ \alpha x^2 + (1 - \alpha)y^2 \pm \sqrt{(1 - 2\alpha)^2 (\alpha x^2 + (1 - \alpha)y^2)^2 + 16 \alpha^2 (1 - \alpha)^2 x^2 y^2} \right]. \quad (47)$$

Figure 3 shows the normalized stresses  $\sigma_{\max} 4a^2/(E A^2)$  and  $\sigma_{\min} 4a^2/(E A^2)$  for the symmetric case  $\alpha = 1/2$ . As a reference, the domain  $\Omega$  of the undistorted membrane has also been plotted at  $z = 0$ . The maximum tensile stress occurs in proximity of the borders, and is directed parallel to them. The minimal tensile stresses are always non-negative and they are equal to zero on the lines  $y = |x|$ . At the corner points the maximum tensile stress is close to the minimal tensile stress and hence, there, the state of stress is sensibly equi-biaxial.

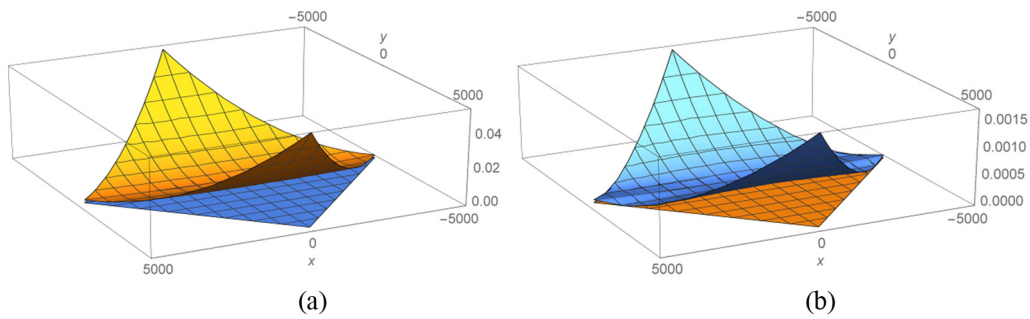
On the other hand, Figure 4 shows the same normalized stress for the case  $\alpha = 0.1$ . To be noted is that the minimal stress is always one order of magnitude less than the maximal stress. The latter is dominant in the  $y$  direction, that is, in the direction along which the membrane is more curved. The minimal stress is non-negative, and it appears to be minimal in a neighbourhood of the line  $y = 0$ , where also  $\sigma_{\max}$  is minimal. The peaks of stress are concentrated in a region around the corners  $(x, y) = (0, \pm a)$ .

For the sake of comparison it may be also interesting to plot in Figure 5 the most severe case that, according to Figure 2, corresponds to  $\alpha = 0.25$ . Both the minimal and the maximal stress grow sensibly, indicating that this configuration should be avoided.

From the analysis of Section 2.2, Proposition 3, if one considers the hyper configuration (30) with  $\alpha \rightarrow 0$  as an approximation of the configuration with zero Gaussian curvature, it should be expected that the maximal tensile forces are directed along the lines parallel to the  $x$ -axis, that is, along the direction of minimal curvature. However, Figures 4 and 5, which correspond to values of  $\alpha$  of the order of 0.1–0.25, evidence that the stress concentrations occurs in a neighbourhood of the corners  $(x, y) = (0,$



**Figure 5.** Normalized stresses (a)  $\sigma_{\max} 4a^2/(E A^2)$  and (b)  $\sigma_{\min} 4a^2/(E A^2)$  for the case  $\alpha = 0.25$ .



**Figure 6.** Normalized stresses (a)  $\sigma_{\max} 4a^2/(E A^2)$  and (b)  $\sigma_{\min} 4a^2/(E A^2)$  for the case  $\alpha = 0.03$ .

$\pm a$ ), that is, in a direction orthogonal to what could have been imagined. This is indeed a characteristic of the hyper configuration, which produces stress concentrations in the proximity of the corner points, in the direction where the fibres are more curved. Of course, to withstand such stresses, in these regions the membrane could be locally reinforced with additional fibres with no practical difficulty. However, in order to obtain a stress state that resembles more strictly the ideal configuration of zero Gaussian curvature, the value of  $\alpha$  needs to be further diminished. To illustrate, Figure 6 represents the counterpart of the previous figures for the case  $\alpha = 0.03$ . The peaks drastically reduce and, apart from these, the state of stress is almost null. This finding is in agreement with the discussion of Section 2.2, according to which the basic state of stress associated with the pure geometric demand is null if the deformed surface is developable.

Of course, the curvature in the  $x$  direction becomes almost zero for very small values of  $\alpha$ , but one can compensate for the consequent loss of out-of-plane stiffness by adding an additional *homogeneous* tension state in the membrane, which in the expressions (46) is associated with the term  $B_4$ . What is important to limit in any case is the peak in the neighbourhoods of the points  $(x,y) = (0, \pm a)$ , because this is associated with the strain capacity required by the membrane to fulfil the geometric compatibility. Since such contribution is governed by the elastic modulus of the material, the corresponding stress increases proportionally to the stiffness of the membrane.

Completely specular considerations hold when  $\frac{1}{2} < \alpha < 1$ . In this case the curvature is the smallest in the  $y$  direction and the maximal stresses occur in a neighbourhood of the points  $(x,y) = (\pm a, 0)$ .

### 3.3 Changing curvatures and the height of the supports at the corners

We have seen in the previous sections that one can reduce the state of stress associated with the geometric demand by remaining close to the limit configurations, that is, when  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 1$ . One can move the membrane between these two states, but in general, in order to avoid the excessive tensioning in the intermediate stages, in particular for  $\alpha = 0.25$  and  $\alpha = 0.75$ , it would be necessary to slacken the membrane while passing from one limit configuration to the other limit configuration. This is certainly

possible, but for many reasons, in particular aesthetics, it would be preferable to produce a smooth transition, so as to render the membrane taut in all the intermediate stages compatible with its strength.

To achieve a full transformable-architectural artwork while limiting the stress in the intermediate stages, we can choose to modify also the parameter  $A$  in (30). In other words, one can mitigate the state of stress associated with the geometric demand in the intermediate configuration by simultaneously reducing the distance between the corner points. In fact, as is clear from (46), the state of stress is directly proportional to  $A^2$ . Of course, this new control represents a complication from a technical point of view, because a mechanism should be employed that allows not only modifying the height of the corner supports, but also the aesthetic/architectural result would be very much improved.

To illustrate this technique, suppose that the parameter  $\alpha$  is modified according to the history  $\alpha(t) = \alpha_0(1-t) + (1-\alpha_0)t$ , where  $\alpha_0 \ll 1$  and  $t \in (0,1)$  is a parameter representative of time. One can likewise modify  $A = A(t)$ , by assuming for example a law of the type

$$A(t) = A_0 \sqrt{\frac{\alpha_0(1-\alpha_0)}{\alpha(t)[1-\alpha(t)]}}, \text{ for } 0 \leq t \leq 1, \tag{48}$$

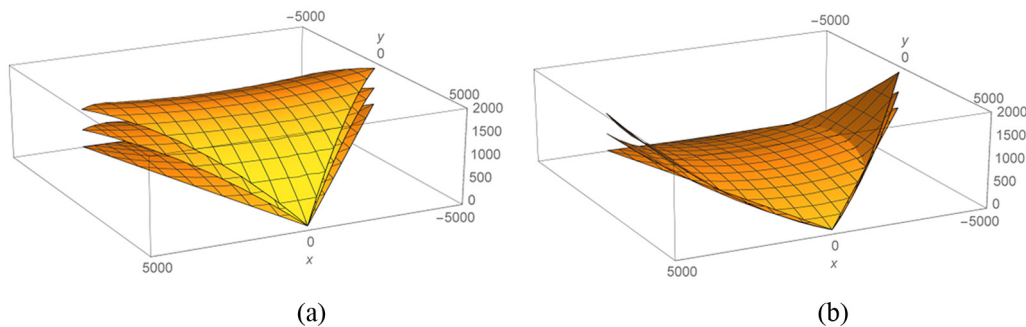
in such a way that the product  $A(t)^2\alpha(t)[1-\alpha(t)]$  remains constant for  $0 \leq t \leq 1$ . For the case  $A_0/2 = 2$  m,  $a = 5$  m,  $\alpha_0 = 0.1$ , the transforming hyperbolic paraboloid is shown in Figure 7(a) for  $t = 0, t = 0.1, t = 0.5$ , and in Figure 7(b) for  $t = 0.5, t = 0.75, t = 1$ . In particular, in Figure 7(a) the points  $(x,y) = (\pm a,0)$  are moving down, whereas in Figure 7(b) they are moving up again, according to the trend of  $A(t)$  represented in Figure 8(a). There is a smooth transition from one configuration to the other.

As is clear from the discussion of Section 3.2, the maximum values of  $\sigma_{\max}$ , given by (47), are attained at  $(x,y) = (0, \pm a)$  for  $0 \leq \alpha \leq 0.5$  and at  $(x,y) = (\pm a, 0)$  for  $0.5 \leq \alpha \leq 1$ . Figure 8(b) represents the corresponding peak values of  $\sigma_{\max}/E$  as a function of  $t$ , calculated for the history (48).

Observe that the stress does not remain constant during the transformation process, but reaches a minimum at  $t = 0.5$ . In order to regularize such a trend, one could imagine to slightly change the law for  $A(t)$ . Assume then the relationship

$$A(t) = \begin{cases} A_0 \sqrt{\frac{\alpha_0(1-\alpha_0)^3}{\alpha(t)[1-\alpha(t)]^3}}, & \text{for } 0 \leq t \leq 0.5, \\ A_0 \sqrt{\frac{\alpha_0(1-\alpha_0)^3}{\alpha(t)^3[1-\alpha(t)]}}, & \text{for } 0.5 \leq t \leq 1, \end{cases} \tag{49}$$

so that the quantities  $A(t)^2\alpha(t)[1-\alpha(t)]^3$  and  $A(t)^2\alpha(t)^3[1-\alpha(t)]$  remain constant for  $0 \leq t \leq 0.5$  and  $0.5 \leq t \leq 1$ , respectively. Figure 9(a) represents the corresponding graph of  $A(t)$ , which is clearly non-monotone and presents its maximum value at  $t = 0.5$ . The corresponding values of the peaks of  $\sigma_{\max}/E$  are drawn in Figure 9(b); remarkably, the maximum stress remains constant throughout the whole process.



**Figure 7.** Transformable hyperbolic paraboloid for (a)  $t = 0, t = 0.1, t = 0.5$  and (b)  $t = 0.5, t = 0.75, t = 1$ . Case  $A/2 = 2$  m,  $a = 5$  m,  $\alpha_0 = 0.1$ .

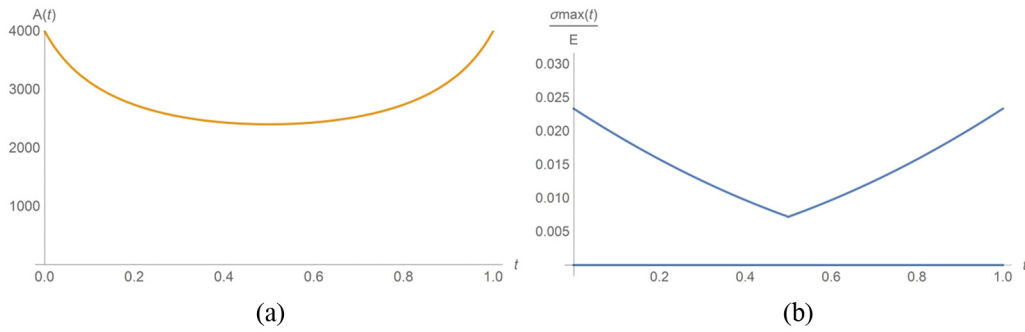


Figure 8. (a) Plot of  $A(t)$  according to (48). (b) Plot of  $\sigma_{\max}/E$  as a function of  $t$ .

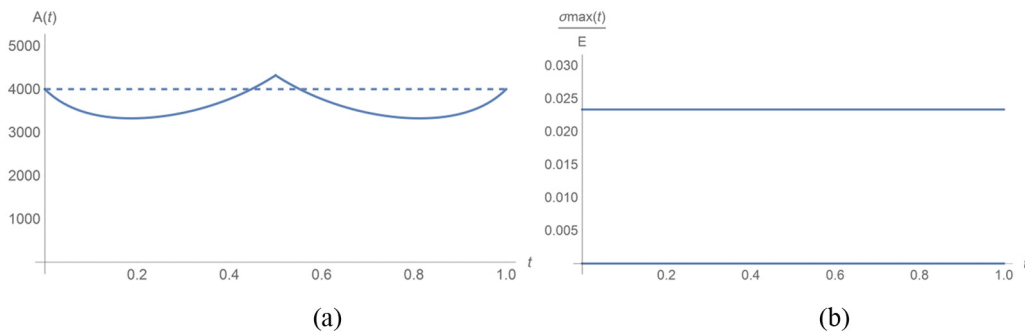


Figure 9. (a) Plot of  $A(t)$  according to (49). (b) Plot of  $\sigma_{\max}/E$  as a function of  $t$ .

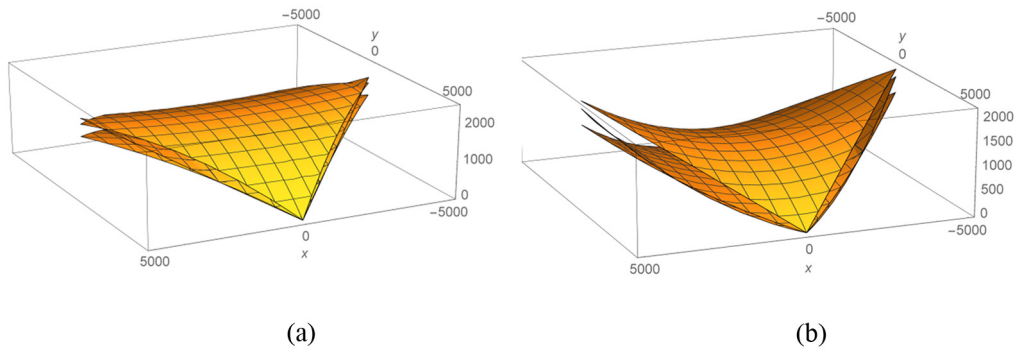
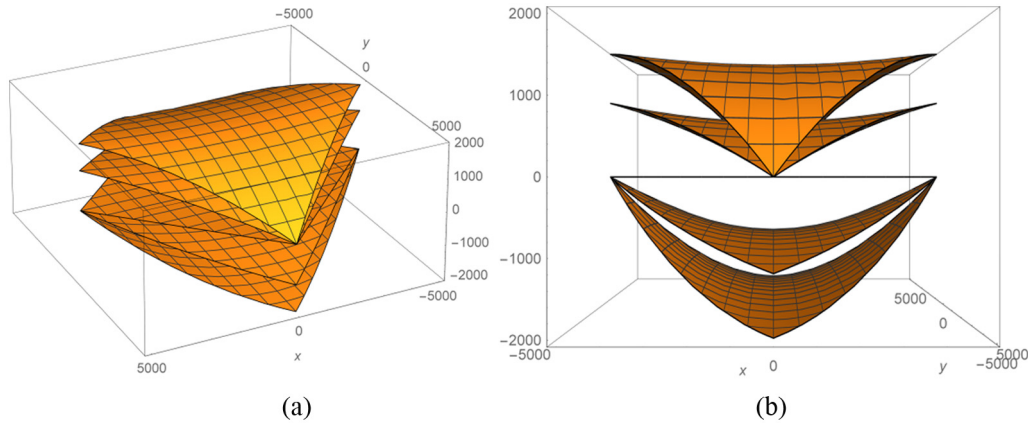


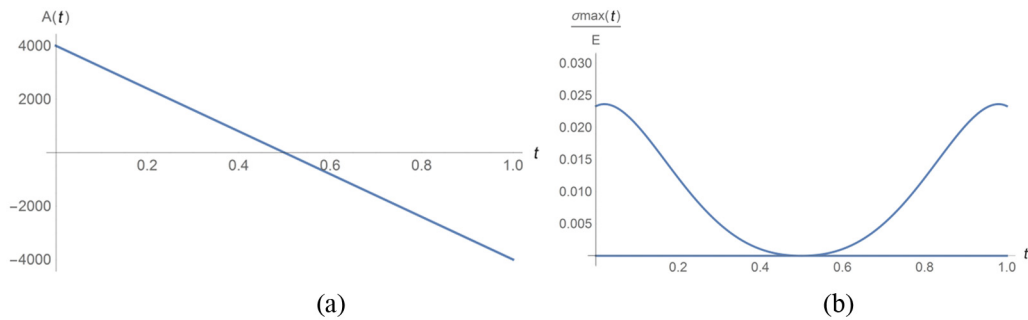
Figure 10. Transformable hyper according to the law (49) for (a)  $t = 0, t = 0.2, t = 0.5$  and (b)  $t = 0.5, t = 0.9, t = 1$ . Case  $A/2 = 2$  m,  $\alpha = 5$  m,  $\alpha_0 = 0.1$ . Intersecting shapes for  $t \cong 0.5$ .

Figure 9(a) evidences the trend of  $A(t)$ , and in particular that the configuration associated with  $t = 0.5$  ( $\alpha = 1/2$ ) is slightly higher than those corresponding to  $t = 0$  or  $t = 1$ . The “intersection” of the various transforming shapes is recorded in Figure 10(a) for  $t = 0, t = 0.2, t = 0.5$ , and in Figure 10(b) for  $t = 0.5, t = 0.9, t = 1$ .

In any case, the advantage of applying this particular type of transformation is null in practice. In fact, observing Figures 3–6, it is evident that in the configuration  $\alpha = 1/2$  the maximum stress affects a great portion of the membrane (Figure 3), whereas when  $\alpha$  is close either to 0 or to 1, only a neighbourhood of the corners has to withstand the peak tensile stress. Consequently, if the membrane is locally reinforced only in these regions, the maximum stress attained in the intermediate configuration  $\alpha = 1/2$  should be kept much smaller than in the limit configurations  $\alpha \cong 0$  or  $\alpha \cong 1$ . This is the reason why the stress variation represented in Figure 8(b) should be preferred to that of Figure 9(b).



**Figure 11.** Transformable hyperbolic paraboloid (two views) according to (50). Configurations at  $t=0, t=0.2, t=0.5, t=0.8, t=1$ . Case  $A/2=2$  m,  $a=5$  m,  $\alpha_0=0.1$ .



**Figure 12.** (a) Plot of  $A(t)$  according to (50). (b) Graph of  $\sigma_{\max}/E$  as a function of  $t$ .

Of course, one can achieve more mobility by allowing for both negative and positive values of  $A$ . In the simplest case, one can consider a linear law for  $A(t)$ . Maintaining the same trend for  $\alpha$ , consider then

$$A(t) = A_0(1 - t) - A_0t, \quad \alpha(t) = \alpha_0(1 - t) + (1 - \alpha_0)t, \quad t \in (0, 1),$$

$$w(t) = \begin{cases} \frac{A(t)}{2a^2} [\alpha(t)x^2 - (1 - \alpha(t))y^2 + (1 - \alpha(t)) a^2], & \text{for } A(t) \geq 0, \\ -\frac{A(t)}{2a^2} [\alpha(t)x^2 - (1 - \alpha(t))y^2 - \alpha(t) a^2], & \text{for } A(t) < 0. \end{cases} \quad (50)$$

The deformations of the membrane corresponding to  $t = 0, t = 0.2, t = 0.5, t = 0.8, t = 1$  are represented in Figure 11. The (linear) law for  $A(t)$  and the corresponding values of the peak value of  $\sigma_{\max}/E$  are drawn as a function of  $t$  in Figure 12.

Of course more elaborated laws for  $A(t)$  could be considered, but in any case this type of transformation needs to pass through an intermediate configuration with  $A(t) = 0$ , which corresponds to a flat membrane. In this configuration, associated in (50) with  $t = 0.5$ , the stress consequent to the geometric demand only is obviously zero, as confirmed by Figure 12(b) that represents the peak value of  $\sigma_{\max}/E$ . In this particular configuration, slackening must be avoided by adding the additional state of stress associated with the term  $B_4$  in (46). Since  $\alpha = 1/2$ , this state will be homogeneous and equi-biaxial.

#### 4. Technical aspects and conclusions

This research has found that transformable tensile membranes can be designed to achieve more than one equilibrium stable state by changing the boundary conditions to meet different geometrical

configurations. The membrane moves through these different states by large deformations, and in doing so it defines a dynamic architectural space.

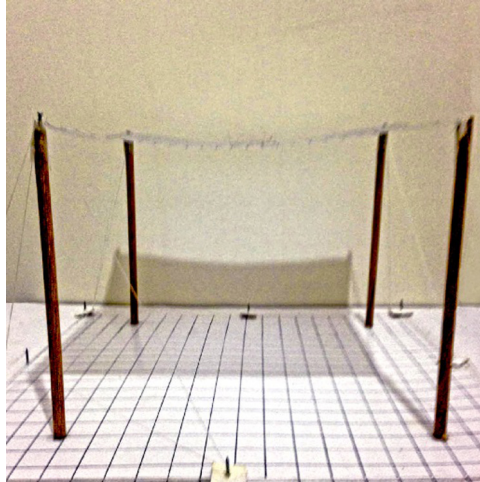
The traditional approach to the design of tensile membranes usually consists of a non-linear form finding process, which is based upon different calculation algorithms: the designer decides the boundary conditions, the type and position of supports, and the prestress that should be given to the membrane, finding a posteriori the shape that is compatible with these conditions. Most often, the target state of stress is chosen to be uniform and equi-biaxial in order to get a stable minimal surface; the initially flat membrane is thus opportunely cut accordingly to fit the corresponding curvature. In our study, we seek various equilibrium configurations, considerably distant from one another. To pursue this goal, the plane and developable surface appears to be the only *neutral* reference configuration from which one can move, up and down, towards the alternating desired shapes.

We have demonstrated (Section 2.2) that, under the hypotheses of the proposed membrane model, the ideal equilibrium of a minimal surface cannot be applied to an initially flat membrane without cutting it. However, clearly this is not a reversible operation. Moving away from the flat referential configuration induces a state of stress that is highly non-uniform: an extra tensioning is therefore asked of the material, but this may be not compatible with its strength.

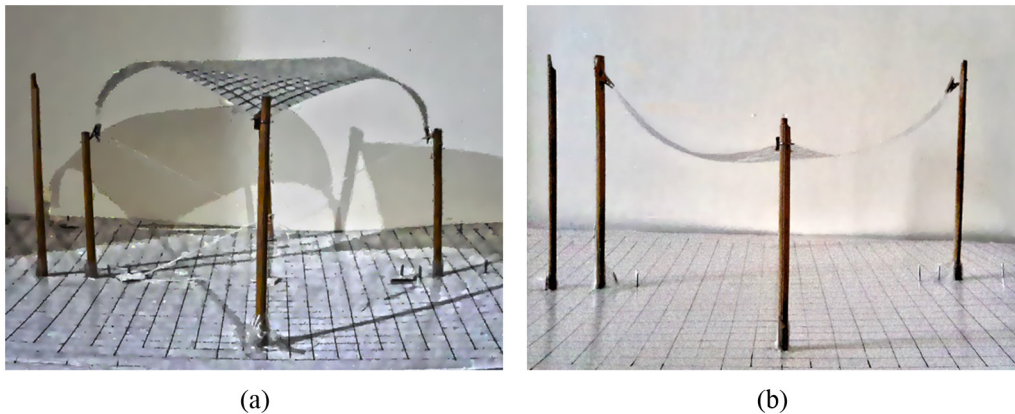
The strategy suggested here is to depart from the ideal condition of equi-biaxial stress, while avoiding compressive stresses that cannot be resisted by the membrane. In particular, we look for homogenous tensile states of stress (not equi-biaxial) at equilibrium, which are certainly associated with a uniform distortion of the membrane (without wrinkling). We have demonstrated in Section 2.2 that the only deformed surfaces associated with states of this kind are characterized by having null Gaussian curvature, that is, they are represented by a developable surface. In particular, we can admit configurations where the tensile principal stress is positive in the direction of null principal curvature and null in the orthogonal direction, where the curvature is maximal in absolute value. Such a deformation is obviously compatible with an undistorted flat state and, by simply changing the boundary conditions, it is possible to transform this configuration into other ones, characterized by another state of stress, similar but “rotated” with respect to the previous one.

However, configurations of this kind would be very flexible with respect to normal loads. Therefore, we need to consider (small) deviations from the aforementioned states that allow tensioning the membrane in both directions, while limiting the geometric demand in terms of strain. In particular, referring to the exemplary hypar-like surfaces, we discuss what is the strain demand associated with deviations from the reference plane, and we furnish criteria for limiting the consequent stress, taking into account also the elasticity of the material. Deformations close to the ideal developable surface condition, where one principal curvature is much higher than the other, are preferable. Yet, if also the vertical movement of the corners is allowed, one can achieve a broader class of large deformations, without producing the slackening of the membrane. Although the method is applicable only to quasi-developable surfaces, this is not a narrow restriction in practice, because developable surfaces and their aggregations still suit most architectural shapes, and even concepts for free-form architectural envelopes can sometimes or in part be made by means of aggregations of developable surfaces without aesthetical detriment [31].

The practical implementation of the movements deduced from this theoretical study presents some basic problems with regard to the material, the supports and the tensioning apparatus. The greatest difficulty certainly consists in conceiving a device that is able to impart to the membrane the required boundary conditions, which are variable according to the desired deformation. Usually, membranes are tensioned by cables sliding through pockets on their edges, or by a rigid supporting frame. For transformable membranes cables should be preferred, because they are flexible and therefore they can better adapt to the moving surface. Of course, according to (8) and (11), it is necessary to distinguish the in-plane and the out-of-plane edge forces. A possible method is to use more than one cable for each border of the membrane, so to distinctly impart the desired components of tensile boundary stress in the warp and fill directions. For example, one cable may achieve a catenary shape in the plane tangent to the membrane, and the second one may form another catenary at an angle with that, so that the stress transmitted to the border has a non-null component in the vertical plane and the vertical displacement can be controlled. By changing the tensile stress in both cables, the desired boundary conditions can, at least approximately, be fulfilled. Although twin cables have been already used to provide concentric



**Figure 13.** The membrane formed by the polyvinyl chloride sheet in the undistorted flat state.

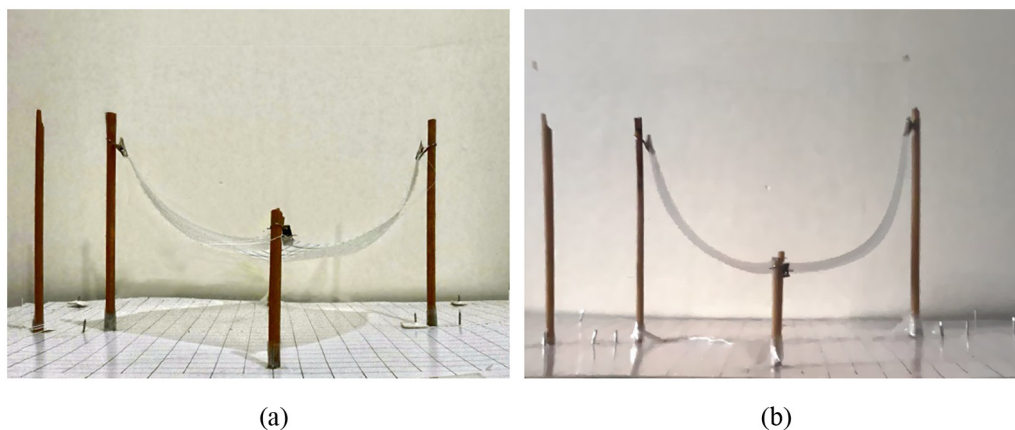


**Figure 14.** The membrane is transformed by tensioning it along either one diagonal or the other. Two different, anti-symmetrical configurations arise while maintain fixed the heights of the corners.

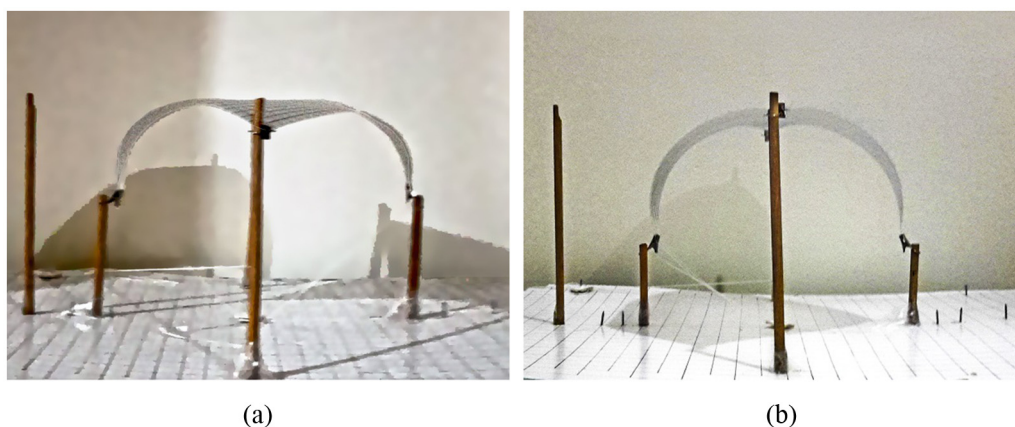
joints, assigning different stress in the warp and fill directions requires new procedures during the manufacturing process and installation.

To clarify the results of this study, a simplified physical model was constructed with a thin polyvinyl chloride (PVC) sheet, with a grid drawn on it for the sake of visual clarity. The undistorted reference configuration of the membrane in its flat configuration is illustrated in Figure 13.

If, as discussed in Section 3.2, the height of the vertexes is kept fixed, the two alternating configurations are those illustrated in Figure 14. These two geometrical deformations are anti-symmetrical, very distant from one another, and in both of them one of the principal curvatures is almost null. The two configurations are obtained by alternatingly tensioning the membrane either along one diagonal or the other one. In the case considered here the membrane is clamped at the corners, so that the stress is transmitted only through these points. This is certainly a very rough approximation of the boundary conditions (8) and (11), but it is tolerable in the physical model because the PVC sheet has a noteworthy stiffness and strength. It is clear from the physical model that the supports need to be moved in the horizontal direction in order to modify the state of stress in the membrane: this can be done by changing the axial forces in the back-stays that support the four masts to which the membrane is attached. In fact, when the radius of curvature in one direction increases, the two vertexes in the same direction get



**Figure 15.** Starting from the configuration of Figure 14(b), the membrane is further deformed by letting the end supports along the direction of the lower curvature move down.



**Figure 16.** From the configuration of Figure 14(a), the membrane is further curved by letting the clamped point in the direction of positive curvature move down.

closer in space, and the contrary occurs when the radius of curvature decreases. In the case of a very flexible membrane this control is certainly more difficult, because the whole sides of the body need to be governed, not only the vertexes. To pass from configuration (a) to configuration (b) of Figure 14, one has to release the membrane and invert the proportions between the principal stresses.

One can reach a smoother transition by allowing for the vertical displacement of the supporting points. In Figure 15, starting from the configuration of Figure 14(b), the membrane is transformed by increasing the tension in the direction of lower curvature and, at the same time, by lowering the position of the supports.

Equivalently, from the configuration of Figure 14(b), the membrane can be moved towards the states represented in Figure 16.

It is clear that these possibilities of transformation can be fully explored just if movable supports are used. The transformable frame should be designed so that it is compatible with the required movements. An example of this kind, specific for the hyper-like structure, is the one proposed in [32].

Concerning the material, in the proposed example (hyperbolic paraboloid) the geometry of the fabric should be diagonally patterned, in agreement with the fact that, in the proposed configurations, the membrane is alternatively tensioned along the warp and fill directions. Observe that, in general, it is possible to reduce the geometric demand in terms of stress for the case of an orthotropic membrane,

because the shear modulus can be tuned to be low enough to reduce the effects due to the angle variation of fibres parallel to the directions of elastic symmetry [33]. This gives noteworthy advantages, especially in the most curved configurations [30].

Furthermore, observe that although the present analysis has been conducted for homogeneous isotropic membranes, the extension to orthotropic membranes presents no difficulty provided that the material is still homogeneous.

This short discussion of the engineering aspect of the proposed design is certainly not exhaustive, but it can highlight the challenges of the implementation process of this kind of transformable structures. Yet, there are knowledge and technologies to provide solutions to these and the other aspects, and it is a design challenge to make membranes light and elegant. Indeed, making structures capable of unusual tasks and additional performances may also demand creative solutions that do not overwhelm the added pros because of a bulky appearance, low reliability or extraordinary cost.

Of course, the hyperbolic paraboloid is just the ordinary example of a tensile membrane. Other transformable tensioned structures can be formulated that reproduce the same concept-design presented here, that is, the approximation of developable surfaces with surfaces whose Gaussian curvature is moderate. Further design opportunities arise if more tensile membranes are considered, one close to the other, since from their deformations new spaces can be created. These transformations are not just aesthetically pleasant, especially if the movement is achieved by means of elegant devices, but also they can let the structure host various functions, or permit to the space the fulfilment of different performances. The wide range of new design opportunities is yet to be fully appreciated.

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## Notes

1. In fact, since  $f_{,xx} = 0$ , then  $f_{,x} = g_1(y)$  and  $f = x g_1(y) + g_2(y)$ . Then  $f_{,yy} = 0$  implies that  $x g_1''(y) + g_2''(y) = 0$  for all  $x$ , which provides the condition that  $g_1''(y) = g_2''(y) = 0$ . Then both  $g_1(y)$  and  $g_2(y)$  are linear functions of  $y$ .
2. This can be demonstrated with an argument similar to that used in note 1.

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