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Parabolic and cubic acceleration time integration schemes for nonlinear structural dynamics problems using the method of weighted residuals

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Abstract

Two algorithms are proposed for direct time integration of equation of motion of structural dynamics problems. The performance of the proposed methods is examined by evaluating stability, order of accuracy, numerical dissipation, and algorithmic damping. The results show that critical time for instability of the proposed algorithms is larger than those of conditionally stable methods. The numerical dissipation is shown to be explicitly less than other methods. Furthermore, the proposed algorithms are non-dissipative in the low-frequency range and have favorable damping in mid and high-frequency regimes. Three examples are carried out to evaluate the feasibility and effectiveness of the proposed algorithms.

Keywords: Time integration, higher order acceleration variation, numerical stability, weighted residual, algorithmic damping, numeric dispersion

Nomenclature

a_i to f_i	:	Unknown integral constants
A	:	Numerical amplification matrix
A_{11} to A_{33}	:	Components of amplification matrix
AD	:	Amplitude decay
c	:	Damping coefficient
f_s	:	Restoring force
m	:	Mass of a system
p	:	External force
PE	:	Period elongation
R	:	Residual
t	:	Time parameter
T	:	Natural period of vibration
$\overline{T}$	:	Numerical period of oscillation
u	:	Displacement of system
$\dot{u}$	:	Velocity of system
$\ddot{u}$	:	Acceleration of system
u_0	:	Initial displacement
$\dot{u}_0$	:	Initial velocity
u_p	:	Displacement particular solution
$\dot{u}_p$	:	Velocity particular solution
W_i	:	Weight function
x_{11} to x_{33}	:	Components of coefficient matrix
α	:	Order of accuracy
α_1	:	Half-trace of amplification matrix
α_2	:	Determinant of amplification matrix
φt	:	Time step size
φt_{cr}	:	Critical step size for stability
λ	:	Eigenvalue of amplification matrix
μ	:	Stiffness ratio
ξ	:	Damping ratio
$\overline{\xi}$	:	Numerical damping
$\rho(A)$	:	Spectral radius
τ	:	Time variable changing from 0 to φt

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ϕ	:	Numerical phase
ω	:	Highest natural cyclic frequency of the system
$\bar{\omega}$	:	Numerical cyclic frequency of oscillation

1. Introduction

It is a fact that most problems in engineering are governed by differential equations. Spatial discretization of engineering structures through, for example, finite difference or finite element methods yields the governing equation of motion as second-order differential equations. Owing to the complexities of loading and nonlinearity of structural systems, analytical solutions are rarely possible. Further discretization of equation of motion of structures is then required and step-by-step integration schemes are used to obtain the time history of response. Many methods exist in the literature and are presented in textbooks by Hughes [1], Wood [2], Zienkiewicz and Taylor [3], Hairer and Wanner [4] and are discussed in review papers by Nickell [5], Dokainish and Subbaraj [6, 7], and Fung [8].

The construction of time step integration schemes via Taylor series collocation methods, Rung-Kutta methods, Weighted Residual Methods, Hamiltonὂs principle, and least square methods [2] have been the focus of considerable attention in the last fifty years [9]. This time period has also witnessed a rapid growth of computational power and a demand for solving large and complex systems. Due to these reasons, along with the inherent simplicity involved in time integration procedures for solving multi-degree-of-freedom structural systems and nonlinear dynamic problems, the development of effective and accurate integration algorithms appears to be the emphasis of future research studies.

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Generally, all step-by-step integration algorithms can be categorized as either explicit or implicit. If the displacements for the next time step can be determined from the accelerations, velocities and displacements of the current time step, without the use of equation of motion at the next time step, the integration scheme is explicit; otherwise it is referred to as implicit. An integration scheme is considered to be stable if the numerical solution under any arbitrary set of initial conditions does not grow without bound [10]. An algorithm is unconditionally stable if the solution is independent on the size of the time step ϕt. It is conditionally stable if the stability of solution depends on a critical value of ϕt expressed as ϕt_{cr}. The value of the critical time step is equal to $\omega\phi t$, in which ω is the highest natural frequency of the system. Explicit algorithms are generally conditionally stable [11] while most implicit methods are unconditionally stable. It is implied in implicit methods that nonlinear iterations are involved at each time step, which is rather time consuming. While the explicit methods contain no nonlinear iterations, the stability requires a small time step size and a very large number of time steps [12]. Apparently, either an implicit method or an explicit method has its own advantages and disadvantages. To ameliorate these deficiencies, semi-implicit integration methods have been proposed by, among others, Hughes et al. [13], Sakai et al. [14], Honda et al. [15], Steerneman [16], and Chang and Huang [17]. The general idea is to discretize differential equations using an implicit scheme for stiff components and an explicit scheme for flexible components of a structural system.

In addition to stability; dissipation error, dispersion error, order of accuracy, overshooting effect, and computational time are other substantial factors that are utilized to evaluate the performance of time integration algorithms [1, 18]. Dispersion (relative period error) indicates numerical errors with elongating or shortening the natural period of vibration in comparison with

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exact quantities, and dissipation (numerical damping) is numerical error with decreasing or increasing the amplitude of vibration in comparison with exact values [19, 20]. The latter is also referred to as amplitude decay and the two are kinds of numerical errors controlling the accuracy of an integration algorithm. Generally, the accuracy also depends on the size of time step φt. An integration scheme is convergent if the numerical solution approaches to the theoretical solution as φt verges to zero. The magnitude of the numerical error is proportional to $(\varphi t/T)^{Ŭ}$, where Ŭ denotes the order of accuracy. According to Dalhquist [11], all unconditionally stable methods are accurate to the second order. In other words, third and higher order accurate unconditionally stable linear multistep algorithms do not exist. The phenomena in which excessively large oscillations in displacement and velocity solutions may occur in the early response and is called an overshooting effect [21]. The tendency to overshoot is an important factor and hence should be studied in the evaluation of an integration algorithm.

It has been recognized that numerical dissipation is important for direct time integration methods. Despite the fact that being regarded as a drawback by some researchers [9, 22], numerical dissipation is considered by many as a necessity for time integration methods to possess [23]. When using standard finite difference or finite elements to discretize a system, the spatial resolution of the domain becomes poor and, consequently, high frequency modes cannot be represented accurately. Therefore, the responses of the spurious high frequency oscillations should be damped out via algorithmic damping. The inclusion of material damping has been shown to affect the middle band of frequencies, not the inaccurate higher frequency modes [24]. Numeric damping can additionally improve the convergence of iterative equation solvers in nonlinear problems [25]. One of the prelusive algorithms to include algorithmic damping is the

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Houlbolt method [26]. The method has also the advantage of asymptotic damping. However, the method has been observed to be very dissipative in the low frequency range. A number of methods have since been developed with an aim to preserve low frequency response while damping high frequency responses in a controllable way. Some of the well-known algorithms include the Newmark method [27], Wilson-ɗ method [28], Park method [29], collocation methods [30], HHT-Ŭ method [24, 30], WBZ-Ŭ method [31], ɟ method [32], and generalized Ŭ methods [21]. Some of these methods are linear multi-step algorithms and usually require additional algorithmic complexity.

An ideal integration scheme is suggested to have the following criteria: at least second order accuracy, unconditional stability in linear problem applications, controllable algorithmic damping, no overshoot, self-starting, single-step, and asymptotic annihilation [1, 19, 33]. From a practical point of view, computational cost, accuracy, stability, parametric damping, design of propagation of information, and type of inertia matrices are the important features that a time integration method should ideally contain. [34]. No single method yet satisfies all the criteria mentioned. It seems that the effort to develop an efficient and accurate numerical integration scheme that satisfies criteria partially or fully will continue to be looked into by researchers.

Many time integration schemes are based on the Taylor series collation method, where it is assumed that the variation of acceleration, velocity, and displacement within each time step follows a particular pattern with a polynomial of a certain degree. It is well known that classical methods such as the Newmark and Wilson-ɗ methods assume a cubic variation of displacement at each time step [8, 35]. Retaining more terms in the Taylor series expansion, higher order polynomials may be utilized for interpolation. Hence, higher accuracy can be achieved. Such an analysis is proposed in this study. A fourth and fifth degree of displacement variations are assumed. Therefore, the order of accuracy of the proposed algorithm is expected to be higher than those of the companion methods. The properties of the algorithm are examined in terms of numerical stability, dispersion, dissipation, and order of accuracy. Finally, numerical examples of both linear and nonlinear problems are carried out to observe and evaluate the performance of the algorithm.

2. Proposed Algorithm

The objective of this paper is to increase the order of accuracy of time integration scheme by increasing the order of presumed response variation at each time step. It is assumed that the variation of displacement function is quartic. Quintic variation of displacement response is also considered. It means the acceleration changes quadratically and cubically with time. The former proposed scheme is an extension of the inspirational work of Gholampour and Ghassemieh [20]. The governing equation of motion for a nonlinear single-degree-of-freedom system is

$$m\ddot{u} + c\dot{u} + f_s(u,\dot{u}) = p(t) \tag{1}$$

where m and c are the mass and damping coefficient, f_s is the nonlinear restoring force, $p(t)$ is the external load as a function of time. u, $\dot{u}$, $\ddot{u}$ are the unknown displacement, velocity, and acceleration responses, respectively. This equation can easily be extended for multi-degree-of-freedom systems. Initial conditions at t=0 are

$$u(0) = u_0 \text{ and } \dot{u}(0) = \dot{u}_0 \tag{2}$$

2.1. Fourth Order Displacement Function

Over the time interval $t_i \le t \le t_{i+1}$, the time variable τ changes from 0 to Δt_i. The displacement function can then be expressed as

$$u(\tau) = a_i\tau^4 + b_i\tau^3 + c_i\tau^2 + d_i\tau + e_i \tag{3}$$

in which a_i to e_i are the unknown coefficients to be determined. Within the same time interval, the velocity and acceleration functions are

$$\dot{u}(\tau) = 4a_i\tau^3 + 3b_i\tau^2 + 2c_i\tau + d_i \tag{4}$$

$$\ddot{u}(\tau) = 12a_i\tau^2 + 6b_i\tau + 2c_i \tag{5}$$

Imposing boundary conditions at τ=0 on Eqs. (3) and (4) yields

$$u_i = e_i, \quad \dot{u}_i = d_i \tag{6}$$

By satisfying the equation of motion at the beginning of the current time step and substituting the values of u_i and $\dot{u}_i$ in Eq. (6), one determines c_i:

$$c_i = \frac{p_i - \left(c \cdot d_i + k \cdot e_i\right)}{2m} \tag{7}$$

The equation of motion is also satisfied at the end of current time step and can be expressed as

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + f_{s,i+1}(u,\dot{u}) = p_{i+1} \tag{8}$$

Substituting the acceleration, velocity, and equivalent restoring force into this equation gives

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$$m\left(12a_i\Delta t^2+6b_i\Delta t+2c_i\right)+c\left(4a_i\Delta t^3+3b_i\Delta t^2+2c_i\Delta t+d_i\right)+f_{s,i}$$
$$+k_i\left[\left(a_i\Delta t^4+b_i\Delta t^3+c_i\Delta t^2+d_i\Delta t+e_i\right)-e_i\right]=p_{i+1} \tag{9}$$

where k_i is the stiffness and $f_{s,i}$ the equivalent spring force at the beginning of the current time step. It is assumed that the stiffness of the dynamic system changes from ith to $(i+1)$th time step. Accordingly, the stiffness can be written as

$$f_{s,i+1}=f_{s,i}+k_i\left(u_{i+1}-u_i\right) \tag{10}$$

where $f_{s,i+1}$ and u_{i+1} are the restoring force and displacement at $\tau=\Delta t$ and $f_{s,i}$ is the restoring force at $\tau=0$. Eq. (9) contains two unknowns: a_i and b_i. Therefore, one more equation is needed. Due to polynomial approximation to the displacement response, the sum of inertial, damping and restoring forces will not be in general balanced by the external loading over the whole time domain. Hence, an error or residue will exist:

$$R(\tau)=m\ddot{u}(\tau)+c\dot{u}(\tau)+f_s(u,\dot{u})-p(\tau),\quad 0\le\tau\le\Delta t \tag{11}$$

The notion of Method of Weighted Residuals is to force the residual to zero in some average sense over the time increment [36]. That is

$$\int_0^{\Delta t}W_i(\tau)R(\tau)d\tau=0 \tag{12}$$

where $W_i(\tau)$ is the weight function. In this study, the continuous summation of all the squared residuals is to be minimized. Accordingly, the weight function is determined to be

$$W_i = \frac{\partial R}{\partial a_i} \quad (\text{or } W_i = \frac{\partial R}{\partial b_i}) \tag{13}$$

In Eq. (11), the variation of external force with time can be assumed to be linear over a sufficiently small interval of time φt. Therefore, $p(\tau)$ is given by

$$p(\tau) = p_i + \frac{\tau}{\Delta t}\left(p_{i+1} - p_i\right) \tag{14}$$

Combining Eqs. (9), (10) and (11), and writing the coefficients of a_i and b_i in matrix form yields

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{Bmatrix} a_i \\ b_i \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \tag{15}$$

where

$$\begin{gathered}
x_{11} = 12m\Delta t^2 + 4c\Delta t^3 + k_n\Delta t^4 \\
x_{12} = 6m\Delta t + 3c\Delta t^2 + k_n\Delta t^3 \\
x_{21} = \frac{1}{9}k_i^2\Delta t^9 + k_i c\Delta t^8 + (\frac{24}{7}k_i m + \frac{16}{7}c^2)\Delta t^7 + 16mc\Delta t^6 + \frac{144}{5}m^2\Delta t^5 \\
x_{22} = \frac{1}{8}\Delta t^4(k_i\Delta t^2 + 4c\Delta t + 12m)^2 \\
y_1 = p_{i+1} - f_{s,i} - (2m + 2c\Delta t + k_i\Delta t^2)c_i - (c + k_i\Delta t)d_i \\
y_2 = I_{i2} - f_{s,i}\left(\frac{1}{5}k_i\Delta t^5 + c\Delta t^4 + 4m\Delta t^3\right) - \left(\frac{1}{7}k_i^2\Delta t^7 + ck_i\Delta t^6 + \frac{8}{5}c^2\Delta t^5 + 8mc\Delta t^4 + 8m^2\Delta t^3\right)c_i \\
-\left(\frac{1}{6}k_i^2\Delta t^6 + k_i c\Delta t^5 + 3mk_i\Delta t^4 + c^2\Delta t^4 + 4mc\Delta t^3\right)d_i
\end{gathered} \tag{16}$$

with

$$I_{i2} = \int_0^{\Delta t}\left(k_i\tau^4 + 4c\tau^3 + 12m\tau^2\right)p(\tau)d\tau \tag{17}$$

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Once the five coefficients have been determined, the response values at the end of *i*th time step can be calculated from Eqs. (3), (4) and (5) with $\tau = \Delta t$.

2.2. Fifth Order Displacement Function

It is now assumed that the displacement function has an order of five:

$$u(\tau) = a_i\tau^5 + b_i\tau^4 + c_i\tau^3 + d_i\tau^2 + e_i\tau + f_i \tag{18}$$

There are six unknown coefficients that should be determined. The two initial values of displacement and velocity, and the equation of motion at the beginning of time step present

$$u_i = f_i, \quad \dot{u}_i = e_i \text{ and } d_i = \frac{p_i - \left(c\cdot\dot{u}_i + k\cdot u_i\right)}{2m} \tag{19}$$

The equation of motion at the end of current time step can be written as

$$\begin{aligned} &m\left(20a_i\Delta t^3 + 12b_i\Delta t^2 + 6c_i\Delta t + 2d_i\right) + c\left(5a_i\Delta t^4 + 4b_i\Delta t^3 + 3c_i\Delta t^2 + 2d_i\Delta t + e_i\right) + f_{s,i} \\ &+ k_i\left[\left(a_i\Delta t^5 + b_i\Delta t^4 + c_i\Delta t^3 + d_i\Delta t^2 + e_i\Delta t + f_i\right) - f_i\right] = p_{i+1} \end{aligned} \tag{20}$$

which contains three constants a_i, b_i and c_i that remain as unknowns. The two other required equations are obtained again through the Method of Weighted Residuals by enforcing the residue to have a zero average over the finite time domain. Hence, the following equations are obtained:

$$\int_0^{\Delta t}\frac{\partial R}{\partial a_i}R(\tau)d\tau = 0 \text{ and } \int_0^{\Delta t}\frac{\partial R}{\partial b_i}R(\tau)d\tau = 0 \tag{21}$$

In these equalities, the weight functions are just the derivatives of the residual with respect to unknown constants.

The three unknown values can be calculated using Eqs. (20) and (21), which are expressed in matrix notation as

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{Bmatrix} a_i \\ b_i \\ c_i \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{22}$$

in which

$$\begin{gathered}
x_{11} = k_i \Delta t^5 + 5c\Delta t^4 + 20m\Delta t^3 \\
x_{12} = k_i \Delta t^4 + 4c\Delta t^3 + 12m\Delta t^2 \\
x_{13} = k_i \Delta t^3 + 3c\Delta t^2 + 6m\Delta t \\
y_1 = p_{i+1} - f_{s,i} - (2m + 2c\Delta t + k_i \Delta t^2) d_i - (c + k_n \Delta t) e_i \\
x_{21} = \frac{1}{11} k_i^2 \Delta t^{11} + k_i c \Delta t^{10} + \frac{1}{9}(40k_i m + 25c^2)\Delta t^9 + 25cm\Delta t^8 + \frac{400}{7} m^2 \Delta t^7 \\
x_{22} = \frac{1}{10}(k_i \Delta t^2 + 5c\Delta t + 20m)^2 \Delta t^6 \\
x_{23} = \frac{1}{9} k_i^2 \Delta t^9 + k_i c \Delta t^8 + \frac{1}{7}(26k_i m + 15c^2)\Delta t^7 + 15cm\Delta t^6 + 24m^2 \Delta t^5 \\
y_2 = I_{i2} - f_{s,i}\left(\frac{1}{6}k_i \Delta t^6 + c\Delta t^5 + 5m\Delta t^4\right) - \left(\frac{1}{8}k_i^2 \Delta t^8 + k_i c\Delta t^7 + \frac{11}{3}k_i m\Delta t^6 + \frac{5}{3}c^2 \Delta t^6 + 10mc\Delta t^5 + 10m^2 \Delta t^4\right) d_i \\
-\left(\frac{1}{7}k_i^2 \Delta t^7 + ck_i \Delta t^6 + 4mk_i \Delta t^5 + c^2 \Delta t^5 + 5mc\Delta t^4\right) e_i \\
x_{31} = \frac{1}{10}(k_i \Delta t^2 + 5c\Delta t + 20m)^2 \Delta t^6 \\
x_{32} = \frac{1}{9} k_i^2 \Delta t^9 + k_i c \Delta t^8 + \frac{1}{7}(24k_i m + 16c^2)\Delta t^7 + 16cm\Delta t^6 + \frac{144}{5} m^2 \Delta t^5 \\
x_{33} = \frac{1}{8}(k_i \Delta t^2 + 4c\Delta t + 12m)^2 \Delta t^4 \\
y_3 = I_{i3} - f_{s,i}\left(\frac{1}{5}k_i \Delta t^5 + c\Delta t^4 + 4m\Delta t^3\right) - \left(\frac{1}{7}k_i^2 \Delta t^7 - 8m^2 \Delta t^3 + \frac{8}{5}c^2 \Delta t^5 + k_i c\Delta t^6 + \frac{14}{5}k_i m\Delta t^5 + 8mc\Delta t^4\right) d_i \\
-\left(\frac{1}{6}k_i^2 \Delta t^6 + ck_i \Delta t^5 + c^2 \Delta t^4 + 3mk_i \Delta t^4 + 4mc\Delta t^3\right) e_i
\end{gathered} \tag{23}$$

with

$$
\begin{aligned}
I_{i2} &= \int_{0}^{\Delta t} \left(k_i \tau^5 + 5c\tau^4 + 20m\tau^3\right) p(\tau)\, d\tau \\
I_{i3} &= \int_{0}^{\Delta t} \left(k_i \tau^4 + 4c\tau^3 + 12m\tau^2\right) p(\tau)\, d\tau
\end{aligned}
\tag{24}
$$

Consequently, the responses at $t=t_{i+1}$ are calculated by utilizing the six coefficients just evaluated.

3. Stability of Proposed Algorithms

Errors are introduced into the numerical solution of a time integration procedure due to truncation or assumed variation of response variables. It is of importance to know the influence of the error introduced at one time step on the computations at the next time step. If the error is prone to grow, the solution becomes uncontrollable. In such a case, the integration procedure is said to be unstable. To investigate the stability of a time-step integration algorithm, it is common practice to relate the displacement and velocity at the end of time step (i.e. when $\tau=\Delta t$) to those at the beginning (i.e. when $\tau=0$) as

$$
\begin{aligned}
\begin{Bmatrix} u_{i+1} \\ \dot{u}_{i+1} \end{Bmatrix} &= \begin{bmatrix} A_{11}(\Delta t) & A_{12}(\Delta t) \\ A_{21}(\Delta t) & A_{22}(\Delta t) \end{bmatrix} \begin{Bmatrix} u_i \\ \dot{u}_i \end{Bmatrix} + \begin{Bmatrix} u_p(\Delta t) \\ \dot{u}_p(\Delta t) \end{Bmatrix} \\
&= [A] \begin{Bmatrix} u_i \\ \dot{u}_i \end{Bmatrix} + \begin{Bmatrix} u_p(\Delta t) \\ \dot{u}_p(\Delta t) \end{Bmatrix}
\end{aligned}
\tag{25}
$$

where $[A]$ is the numerical amplification matrix and u_p and $\dot{u}_p$ are the particular solutions related to external load function. The algorithmic properties are determined by the numerical amplification matrix. It is common practice to ignore the particular solution when investigating the stability characteristics of a numeric integration procedure. The reason for this is that if a solution is unstable under conditions of free vibration, it is also likely to be unstable under the conditions of forced vibration. Considering Eqs. (3) and (4), the elements of the amplification matrix can be determined as

$$\begin{aligned}
A_{11} = (&5ck_i^3\Delta t^7 + 65mk_i^3\Delta t^6 + 204mck_i^2\Delta t^5 - 144c^3k_i\Delta t^5 - 276m^2k_i^2\Delta t^4 - 1584mc^2k_i\Delta t^4 - \\
&14832m^2ck_i\Delta t^3 + 2880mc^3\Delta t^3 - 42768m^3k_i\Delta t^2 + 25920m^2c^2\Delta t^2 + 84672m^3c\Delta t + 108864m^4)/D_1 \\
A_{12} = (&5c^2k_i^2\Delta t^6 + 30mck_i^2\Delta t^5 - 420m^2k_i^2\Delta t^4 + 288mc^2k_i\Delta t^4 - 144c^4\Delta t^4 - 1512m^2ck_i\Delta t^3 \\
&- 504mc^3\Delta t^3 - 6480m^3k_i\Delta t^2 + 1728m^2c^2\Delta t^2 + 30240m^3c\Delta t + 108864m^4)\Delta t/D_1 \\
A_{21} = -(&-k_i\Delta t^2 + 12m)(-5k_i^2\Delta t^4 + 216mk_i\Delta t^2 + 144c^2\Delta t^2 + 2520mc\Delta t + 9072m^2)k_i\Delta t/D_1 \\
A_{22} = (&-5ck_i^3\Delta t^7 + 35mk_i^3\Delta t^6 + 192mck_i^2\Delta t^5 + 144c^3k_i\Delta t^5 - 780m^2k_i^2\Delta t^4 + 1440mc^2k_i\Delta t^4 \\
&- 8352m^2ck_i\Delta t^3 + 1152mc^3\Delta t^3 - 42768m^3k_i\Delta t^2 - 4320m^2c^2\Delta t^2 - 24192m^3c\Delta t + 108864m^4)/D_1
\end{aligned} \tag{26}$$

in which

$$\begin{aligned}
D_1 = (&35k_i^3\Delta t^6 + 420ck_i^2\Delta t^5 + 1020mk_i^2\Delta t^4 + 1800c^2k_i\Delta t^4 + 9360mck_i\Delta t^3 + 2880c^3\Delta t^3 \\
&+ 11664m^2k_i\Delta t^2 + 25920mc^2\Delta t^2 + 84672m^2c\Delta t + 108864m^3)m
\end{aligned} \tag{27}$$

The eigenvalues λ of $[A]$ are computed by solving the characteristic equation

$$\left\| [A] - \lambda [I] \right\| = 0 \tag{28}$$

The expansion of this equation gives

$$\lambda^2 - 2\alpha_1\lambda + \alpha_2 = 0 \tag{29}$$

where α_1 is the half-trace and α_2 is the determinant of $[A]$ [37]. They are given by

$$\begin{aligned}\alpha_1 &= 2(25k_i^3\Delta t^6 + 99ck_i^2\Delta t^5 - 264mk_i^2\Delta t^4 - 36c^2k_i\Delta t^4 - 5796mck_i\Delta t^3 + 1008c^3\Delta t^3 \\ &\quad - 21384m^2k_i\Delta t^2 + 5400mc^2\Delta t^2 + 15120m^2c\Delta t + 54432m^3)/D_2 \\ \alpha_2 &= (5k_i^3\Delta t^6 - 24ck_i^2\Delta t^5 - 216c^2k_i\Delta t^4 + 1152c^3\Delta t^3 + 108864m^3 + \\ &\quad (11664k_i\Delta t^2 - 24192c\Delta t)m^2 + (516k_i^2\Delta t^4 - 2304ck_i\Delta t^3 - 4320c^2\Delta t^2)m)/D_2\end{aligned} \tag{30}$$

with

$$\begin{aligned}D_2 &= 35k_i^3\Delta t^6 + 420ck_i^2\Delta t^5 + 1020mk_i^2\Delta t^4 + 1800c^2k_i\Delta t^4 + 9360mck_i\Delta t^3 + 2880c^3\Delta t^3 \\ &\quad + 11664m^2k_i\Delta t^2 + 25920mc^2\Delta t^2 + 84672m^2c\Delta t + 108864m^3\end{aligned} \tag{31}$$

The amplification matrix and the characteristic equation corresponding to the fifth order displacement variation are supplied in the appendix due to the length of the expressions. Solution of Eq. (29) will give two different values of λ denoted by λ_1 and λ_2. The spectral radius of matrix of $[A]$ is defined by

$$\rho([A]) = \max\{|\lambda_1|,\ |\lambda_2|\} \tag{32}$$

An integration procedure is stable if the spectral radius is less than or equal to one. Considering Eqs. (30) and (31), it is evident that the spectral radius is a function of mass, damping coefficient, current state of stiffness, and size of time step. With $\omega = \sqrt{k/m}$, $c=2m\omega\xi$, $\mu_i=k_i/k$, the spectral radius is dependent on damping ratio, stiffness ratio, and the multiplication of cyclic frequency and time step size. The stiffness ratio μ_i is used to incorporate the nonlinear behavior of the system into the solution and is defined as the ratio of stiffness at the beginning of ith time step to the initial stiffness. A plot of spectral radius against $\Delta t/T$ (or $\omega\Delta t$) would display the stability characteristics of the proposed algorithm. For a comparative study, the maximum magnitude of the eigenvalues of methods of Gholampour and Ghassemieh [20], central difference, Newmarkôs methods, Wilson-ɗ method, and the fourth and fifth order methods proposed in this study are plotted. Fig. 1 displays the spectral radius curves for a stiffness ratio of 0.5. For this value, the central difference method with φt>0.45*T* and Newmarkôs linear acceleration method with φt>0.76*T* become unstable. The spectral radius is always equal to unit for Newmarkôs average acceleration method and it is always less than unit for the Wilson-ɗ method, indicating that they are unconditionally stable for a system of nonlinear material. The method of Gholampour and Ghassemieh has a region of local instability for 0.72*T* Ò φt Ò0.77*T*, after which it regains stability. It becomes unstable once φt reaches 1.74*T*. The proposed method, in which the displacement variation is fourth degree, shows a similar behavior to the method of Gholampour and Ghassemieh but the local instability region is much smaller and the critical value of φt/*T* is 1.88. The second proposed scheme with a fifth degree of displacement variation shows no local instability and the critical time for instability is further delayed to φt=3.86*T*. Fig. 2 displays the spectral radius versus φt/*T* for a stiffness ratio of one, which is an indication of an ideally linear elastic material. It is observed from this figure that the critical times for conditionally stable methods are now shorter whereas the unconditionally stable methods are not affected. Only, the eigenvalues of Wilson-ɗ solution dips to a minimum at φt=2.29*T* while the corresponding value for μ_i=0.5 was φt=3.16*T*, at which step the eigenvalues take two distinct real values. Gholampour and Ghassemieh [20] noted that an inclusion of damping, for example ξ=4.6% for a linear system, can remove the local instability occurring in their procedure. It would be of interest to see the influence of damping on the solution algorithms proposed in this study. Fig. 3 shows that the local instability is not an issue for the algorithm with fifth order

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displacement function regardless of the material being linear or nonlinear. The proposed method with fourth degree displacement function is barely affected by the local instability for linear system. An inclusion of slight damping of 1.4% clears the local instability zone. The results presented suggest that the proposed methods, specifically the method with fifth degree of displacement field, have a wider interval of stability than the other conditionally stable methods.

4. Accuracy of Proposed Algorithms

After examining the stability of an algorithm, it is crucial to investigate its accuracy. There are two kinds of numerical errors, i.e. numerical dissipation and numerical dispersion, and order of exactness controlling the accuracy of an algorithm. As the differential equation of motion needs to have an oscillating solution, the eigenvalues of the characteristic equation must be complex conjugates of each other [37]. They are given by

$$\lambda_{1,2} = \alpha_1 \pm \sqrt{\alpha_1^2 - \alpha_2} \tag{33}$$

Note that α_1^2 must be less than α_2 for the mentioned condition to be satisfied. Eq. (33) can be expressed as

$$\lambda_{1,2} = \sqrt{\alpha_2}\left(\cos\phi \pm i \sin\phi\right) \tag{34}$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{\alpha_2 - \alpha_1^2}}{\alpha_1}\right) \tag{35}$$

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In this equation, ϕ is referred to as numerical phase which is related to frequency of solution as

$$\phi = \bar{\omega}\Delta t \tag{36}$$

where $\bar{\omega}$ is the numerical frequency. The period of numerical solution is given by $\bar{T} = 2\pi / \bar{\omega}$. This value will in general be slightly different from the exact period $T = 2\pi / \omega$. As measures of relative accuracy, period elongation (PE) and amplitude decay (AD) is therefore defined as follows:

$$\mathrm{PE} = \frac{\bar{T} - T}{T} = \frac{\omega\Delta t - \phi}{\phi} \tag{37}$$

and

$$\mathrm{AD} = 1 - \alpha_2^{\pi/\phi} \tag{38}$$

As an alternative, AD can also be described by an equivalent viscous damping coefficient such that

$$\bar{\xi} = -\frac{\ln\sqrt{\alpha_2}}{\phi} \tag{39}$$

The period elongation for the considered methods is plotted in Fig. 4 as a function of φt/T. The evaluation of stability indicated that the stiffness ratio (namely, the material of the system being linear or nonlinear) makes the integration procedures less restrictive. Therefore, it is no longer considered in the accuracy assessment of the numeric procedures. Fig. 4 shows that the central difference method will decrease the period while the rest of methods will increase the period of true oscillation. The period elongation of Wilson-θ is the highest. The dispersion errors of Gholampour and Ghassemieh's method and the fourth order proposed method are close to each other. In terms of period elongation, the fifth order proposed method appears to be superior to all other examined schemes.

Numerical damping values of the aforementioned schemes are compared in Fig. 5. It is realized that from Eq. (39) that the algorithmic damping is positive for $\sqrt{\alpha_2} < 1$ and negative for $\sqrt{\alpha_2} > 1$. If $\sqrt{\alpha_2} = 1$, then no algorithmic damping incurs. Ideally, the algorithmic damping should be zero [23]. As stated earlier, however, it is desirable to have some numerical dissipation in the high-frequency region that would damp out responses of spurious high-frequency modes. Also, it has been found that algorithmic damping is helpful in convergence of nonlinear problems in the high-frequency range and in solving problems related to constraints such as contact cases [8]. The spectral radius can also be utilized as a measure of numerical damping. Wood [2] suggests that the optimum dissipative behavior is obtained when the spectral radius stays close to unit in the low- and mid-frequency range and decreases to about 0.5-0.8 in the high frequency range. Fig. 5 illustrates the algorithmic damping plots versus $\Delta t/T$. It is observed that Newmark's average acceleration has zero damping whilst Newmark's linear acceleration and central difference methods have negative dissipation in the middle frequency range. Due to local instability occurring in Gholampour and Ghassemieh's method, a negative dissipation also takes place in that region of the method. The fourth and fifth order scheme proposed in this study has little dissipation in the mid-frequency zone. The amount of damping appears to grow as $\Delta t/T$ increases. The Wilson-θ method has a very favorable amount of numerical damping in the high frequency range; however, the method is rather dissipative in the low- and mid-frequency regions.

The orders of convergence of the proposed schemes are investigated in accordance with a procedure given by Ravazi et al. [18]. The displacement error in the numerical methods is calculated for a system under harmonic excitation using various sizes of time increments. Then, by applying regression analysis to the displacement error, the power of the ratio $\Delta t/T$ is determined. The power is referred to as order of accuracy. Fig. 6 shows the orders of accuracy for the proposed methods against the ratio of ω_0/ω, where ω_0 and ω are the cyclic frequencies of the external load and system, respectively. It is seen that the method with a displacement variation of fourth degree has an order of convergence of more than 4 and the method with a displacement variation of fifth degree more than 5. Commonly used Newmark's methods have an order of about two.

5. Numerical Examples

As the previous stability and accuracy analyses are restricted to the behavior of single-degree-of freedom systems, it is critical that a series of numerical examinations be performed in order to evaluate the properties of the proposed methods in calculating the response of realistic linear or nonlinear multi-degree-of-freedom systems. Therefore, three classical problems are studied. The selection of numerical problems allows the characteristics of the schemes to be readily observed and evaluated.

Example 1. A second-order nonlinear differential equation of motion for a single-degree-of-freedom system given below

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$$\ddot{u}+5\dot{u}+u^3+2000u=400\cos(50t)-100\sin(30t) \tag{40}$$

is considered first. The system has initial the conditions of $u_0=0$ and $\dot{u}_0=0$ and is subjected to an external force having two distinct frequencies. The displacement response of the nonlinear system is obtained using the previous methods. The solution from the Newmarkôs linear acceleration method with φt=0.001 sec can be assumed as ñexactò [35]. The time step size utilized for the other methods is 0.02 sec. It is seen from Fig. 7 that the proposed methods and Newmarkôs average acceleration method closely follow the exact solution while the central difference and Wilson-ɗ show some discrepancies with respect to exact response. Table 1 also presents the data of the displacement response obtained using the integration methods and their errors in predicting the exact solution at certain time intervals. The error is expressed in terms of absolute value of the difference between the two responses. It is observed from the table that the differences between the solutions of the proposed methods and the Newmarkôs linear acceleration method with φt=0.001 sec are the smallest.

Example 2. A two-story shear building is considered in this example. It is assumed that all of the building mass is lumped at the floor levels, that the floor beams are rigid, and that the columns are axially rigid and have varying lateral stiffness. The analytical model of the building is shown in Fig. 8. The mass values and damping coefficients are assumed to be the same for both stories. The positive terms in the story stiffnesses mean that the stories may show hardening behavior. The building is excited by a harmonic loading of the same magnitude but opposite direction at the first and second floor levels. The loading contains two distinctive vibrational frequencies.

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The floor level displacements are determined using the same methods and are presented in Figs. 9 and 10. Again, the linear acceleration method with Δt=0.001 sec is assumed to result in the exact displacement time history. It is observed from these two figures that the central difference method is again erroneous in predicting the exact responses in the whole time history. The Wilson-θ method follows the exact solution in the beginning of analysis but it starts to under or overestimate the exact solution after one second. The response solutions of the proposed fifth degree, fourth degree, and average acceleration method are the closest to the exact floor displacements in that order. This observation is also confirmed by examining the numeric values of displacement response of the first story in Table 2.

Example 3. In this example, an *n* degree-of-freedom damped mass-spring system is considered. The system shown in Fig. 11 is a modification of the problem exercised by Gholampour et al. [35] and Chang [38]. The system has a uniform mass of m_i=10 kg and stiffness coefficient of c_i=5 N·sec/m. The springs have a stiffness of $k_i = 100 - 0.02\sqrt{|u_i|}$ N/mm, where i=1, 2,…,n, and where n denotes the number of degrees of freedom. The forced vibration response of the system is computed through the considered schemes under $p_1 = 100\cos(20t) + 50\sin(30t)$ N applied at the first mass.

Fig. 12 shows the forced response of the first mass in a system with n=100. Only, a two-second portion of the time history of displacement response is plotted. According to the highest natural frequency of the system, a time step duration of Δt=0.001 sec is considered to produce exact results in Newmark's linear acceleration method. A time step size of Δt=0.02 sec is utilized for the remaining methods. Comparing the results obtained from various methods in Fig. 12

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indicates that the results of the considered methods, except for the central difference, closely follow the exact solution. An error analysis shows that the maximum relative error is about 3.5%. The amount of error is in general less in the earlier stages of time history, increases to the maximum at around 1 sec, and remains the same until the end of the analysis.

6. Conclusions

Two procedures are formulized for the direct time integration of equation of motion for linear, nonlinear single, or multi-degree-of-freedom systems. The procedures are based on the fourth and fifth order variation of displacement function with respect to time. This treatment results in the acceleration variation of second and third degree, respectively. The difference between the proposed method of this study and other methods that depend on second-order acceleration function is that no difference equations are resorted to in this study. This adoption presents more unknown coefficients. The unknowns are determined using a) the two initial conditions, b) two equations of motions that are satisfied at the beginning and end of each time step, and c) the integral sum of the weighted residuals. The stability and accuracy of the developed schemes are performed by the use of characteristics of numerical amplification matrix. Analytical examples are also carried out to observe and evaluate the overall behavior of the proposed methods in comparison with other methods, such as the central difference method, Newmarkǿs methods, Wilson-ɗ method, and Gholampour and Ghassemiehǿs method. The selection of these methods includes conditionally and unconditionally stable methods as well as dissipative and non-dissipative methods. As a result of numerical studies, the following conclusions can be drawn:

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i) The proposed methods are only conditionally stable. However, the critical time step is larger than those of other conditionally stable methods: central difference, Newmarkôs linear acceleration, and Gholampour and Ghassemiehôs methods.

ii) The inclusion of material nonlinearity alleviates the restriction on the critical time step size.

iii) The inclusion of material damping also makes the critical time step size larger.

iv) The central difference method shortens the period of exact oscillation while the other methods are expected to increase the true period. The period elongation of Wilson-ɗ is the highest and that of the proposed method with fifth order displacement variation is the least.

v) Newmarkôs average acceleration method has zero numeric damping while Newmarkôs linear acceleration and the central difference methods have negative damping. The Wilson-ɗ method has a favorable amount of numerical dissipation in the high frequency range; however, the method is very dissipative in the low- and mid-frequency regions. The proposed methods have little numeric damping in the middle-frequency region. The amounts are expected to be high in the high-frequency range.

vi) The proposed method with a fourth order displacement variation has an order of more than 4 while the method with a fifth displacement variation has an accuracy order of more than 5. The order of accuracy of the proposed methods, therefore, is larger than the second order that the other classical methods possess.

vii) The numerical examples carried out show that the proposed methods yield response solutions very close to the Newmarkôs linear acceleration method where a rather small time step size is used to simulate the exact response.

In conclusion, the two procedures proposed in this study can be used for the direct time integration of equation of motion of a linear or nonlinear, single or multi-degree-of-freedom system. Further study is required to examine the applicability of the proposed methods in various applications such as a system involving both flexible and stiff components and a system undergoing very large deformations over long time periods.

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Appendix

The components of the amplification matrix for the method in which the displacement variation is fifth order are given by

$$\begin{aligned} A_{11} = 1/8(&21ck_i^5\Delta t^{11} + 504mk_i^5\Delta t^{10} + 1552mck_i^4\Delta t^9 - 1168c^3ki^3\Delta t^9 - 26336m^2k_i^4\Delta t^8 - 25696mc^2k_i^3\Delta t^8 \\ &- 395040m^2ck_i^3\Delta t^7 - 103800mc^3k_i^3\Delta t^7 + 39600c^5ki\Delta t^7 - 844800m^3k_i^3\Delta t^6 - 1632480m^2c^2k_i^2\Delta t^6 \\ &+ 792000mc^4ki\Delta t^6 - 9771840m^3ck_i^2\Delta t^5 + 15428160m^2c^3k_i\Delta t^5 - 1980000mc^5\Delta t^5 - 29168640m^4k_i^2\Delta t^4 \\ &+ 157956480m^3c^2k_i\Delta t^4 - 35640000m^2c^4\Delta t^4 + 787311360m^4ck_i\Delta t^3 - 289872000m^3c^3\Delta t^3 \\ &+ 1633167360m^5k_i\Delta t^2 - 1311552000m^4c^2\Delta t^2 - 3353011200m^5c\Delta t - 3832012800m^6) / D_1 \end{aligned} \quad \text{(A1)}$$

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$$
\begin{aligned}
A_{12} = {} & -1/8\,\Delta t(21c^2k_i^4\Delta t^{10} + 280mck_i^4\Delta t^9 - 5152m^2k_i^4\Delta t^8 + 2336mc^2k_i^3\Delta t^8 - 1168c^4ki^2\Delta t^8 - 36288m^2ck_i^3\Delta t^7 \\
& - 12096mc^3k_i^2\Delta t^7 + 13440m^3k_i^3\Delta t^6 - 240m^2c^2k_i^2\Delta t^6 - 158400mc^4k_i\Delta t^6 + 39600c^6\Delta t^6 + 1897920m^3ck_i^2\Delta t^5 \\
& - 823680m^2c^3k_i\Delta t^5 + 316800mc^5\Delta t^5 + 4285440m^4k_i^2\Delta t^4 + 14446080m^3c^2k_i\Delta t^4 - 1520640m^2c^4\Delta t^4 \\
& + 130394880m^4ck_i\Delta t^3 - 33264000m^3c^3\Delta t^3 + 355829760m^5k_i\Delta t^2 - 273715200m^4c^2\Delta t^2 \\
& - 1437004800m^5c\Delta t - 3832012800m^6)\,/\,D_1
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
A_{21} = {} & 1/8\,\Delta tk_i(21k_i^5\Delta t^{10} - 5084mk_i^4\Delta t^8 - 1168c^2k_i^3\Delta t^8 - 60480mck_i^3\Delta t^7 + 21360m^2k_i^3\Delta t^6 - 206160mc^2k_i^2\Delta t^6 \\
& + 39600c^4k_i\Delta t^6 + 947520m^2ck_i^2\Delta t^5 + 443520mc^3k_i\Delta t^5 + 3430080m^3k_i^2\Delta t^4 + 17012160m^2c^2k_i\Delta t^4 \\
& - 2376000mc^4\Delta t^4 + 130394880m^3ck_i\Delta t^3 - 33264000m^2c^3\Delta t^3 + 355829760m^4k_i\Delta t^2 - 273715200m^3c^2\Delta t^2 \\
& - 1437004800m^4c\Delta t - 3832012800m^5)\,/\,D_1
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
A_{22} = {} & 1/8(21ck_i^5\Delta t^{11} - 224mk_i^5\Delta t^{10} - 4300mck_i^4\Delta t^9 - 1168c^3k_i^3\Delta t^9 + 38432m^2k_i^4\Delta t^8 - 46880mc^2k_i^3\Delta t^8 \\
& + 503520m^2ck_i^3\Delta t^7 - 260760mc^3k_i^2\Delta t^7 + 39600c^5k_i\Delta t^7 + 1161600m^3k_i^3\Delta t^6 + 1629600m^2c^2k_i^2\Delta t^6 \\
& - 31680mc^4k_i\Delta t^6 + 11491200m^3ck_i^2\Delta t^5 + 2439360m^2c^3k_i\Delta t^5 - 396000mc^5\Delta t^5 + 29168640m^4k_i^2\Delta t^4 \\
& - 27561600m^3c^2k_i\Delta t^4 + 2376000m^2c^4\Delta t^4 - 431481600m^4ck_i\Delta t^3 + 16156800m^3c^3\Delta t^3 - 1633167360m^5k_i\Delta t^2 \\
& - 125452800m^4c^2\Delta t^2 - 479001600m^5c\Delta t + 3832012800m^6)\,/\,D_1
\end{aligned} \tag{A4}
$$

where

$$
\begin{aligned}
D_1 = {} & (49k_i^5\Delta t^{10} + 1127ck_i^4\Delta t^9 + 2044mk_i^4\Delta t^8 + 11375c^2k_i^3\Delta t^8 + 51450mck_i^3\Delta t^7 + 62475c^3ki^2\Delta t^7 \\
& + 75360m^2k_i^3\Delta t^6 + 497100mc^2k_i^2\Delta t^6 + 188100c^4k_i\Delta t^6 + 1506600m^2ck_i^2\Delta t^5 + 2336400mc^3k_i\Delta t^5 \\
& + 247500c^5\Delta t^5 + 1365120m^3k_i^2\Delta t^4 + 12331440m^2c^2k_i\Delta t^4 + 4455000mc^4\Delta t^4 + 31315680m^3ck_i\Delta t^3 \\
& + 36234000m^2c^3\Delta t^3 + 35354880m^4k_i\Delta t^2 + 163944000m^3c^2\Delta t^2 + 419126400m^4c\Delta t \\
& + 479001600m^5)\,/\,m
\end{aligned}
$$

(A5)

The expression α_1 and α_2 leading to eigenvalues of the proposed algorithm are as follows

$$
\begin{aligned}
\alpha_1 = {} & 1/2(-182k_i^5\Delta t^{10} - 1463ck_i^4\Delta t^9 + 16192mk_i^4\Delta t^8 - 5296c^2k_i^3\Delta t^8 + 224640mck_i^3\Delta t^7 - 39240c^3k_i^2\Delta t^7 \\
& + 501600m^2k_i^3\Delta t^6 + 815520mc^2k_i^2\Delta t^6 - 205920c^4k_i\Delta t^6 + 5315760m^2ck_i^2\Delta t^5 - 3247200mc^3k_i\Delta t^5 \\
& + 396000c^5\Delta t^5 + 14584320m^3k_i^2\Delta t^4 - 46379520m^2c^2k_i\Delta t^4 + 9504000mc^4\Delta t^4 \\
& - 304698240m^3ck_i\Delta t^3 + 76507200m^2c^3\Delta t^3 - 816583680m^4k_i\Delta t^2 + 296524800m^3c^2\Delta t^2 \\
& + 718502400m^4c\Delta t + 1916006400m^5)\,/\,D_2
\end{aligned} \tag{A6}
$$

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$$\alpha_2 = (3k_i^5\Delta t^{10} - 21ck_i^4\Delta t^9 - 266c^2k_i^3\Delta t^8 + 2130c^3k_i^2\Delta t^7 + 11880c^4k_i\Delta t^6 - 99000c^5\Delta t^5 + 958003200m^5 + (70709760k_i\Delta t^2 - 119750400c\Delta t)m^4 + (2730240k_i^2\Delta t^4 - 8078400ck_i\Delta t^3 - 31363200c^2\Delta t^2)m^3 + (63600k_i^3\Delta t^6 - 358560ck_i^2\Delta t^5 - 2613600c^2k_i\Delta t^4 + 4039200c^3\Delta t^3)m^2 + (644k_i^4\Delta t^8 - 4260ck_i^3\Delta t^7 - 59880c^2k_i^2\Delta t^6 + 396000c^3k_i\Delta t^5 + 594000c^4\Delta t^4)m) / D_2 \quad (A7)$$

with

$$D_2 = (98k_i^5\Delta t^{10} + 2254ck_i^4\Delta t^9 + 4088mk_i^4\Delta t^8 + 22750c^2k_i^3\Delta t^8 + 102900mck_i^3\Delta t^7 + 124950c^3k_i^2\Delta t^7 + 150720m^2k_i^3\Delta t^6 + 994200mc^2k_i^2\Delta t^6 + 376200c^4k_i\Delta t^6 + 3013200m^2ck_i^2\Delta t^5 + 4672800mc^3k_i\Delta t^5 + 495000c^5\Delta t^5 + 2730240m^3k_i^2\Delta t^4 + 24662880m^2c^2k_i\Delta t^4 + 8910000mc^4\Delta t^4 + 62631360 * m^3ck_i\Delta t^3 + 72468000m^2c^3\Delta t^3 + 70709760m^4k_i\Delta t^2 + 327888000m^3c^2\Delta t^2 + 838252800m^4c\Delta t + 958003200m^5) \quad (A8)$$

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Table 1 The displacement response of nonlinear SDOF system obtained from various methods and the error values

	Displacement (m)						Error				
t (sec)	Central Diff	Newmark Ave.	Wilson-ɗ	Proposed 4th	Proposed 5th	Exact	Central Diff	Newmark Ave.	Wilson-ɗ	Proposed 4th	Proposed 5th
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-	-	-	-	-
0.25	0.0148	-0.0070	-0.0086	-0.0072	-0.0073	-0.0076	0.0224	0.0006	0.0010	0.0004	0.0003
0.50	-0.0280	-0.0160	-0.0165	-0.0165	-0.0166	-0.0167	0.0113	0.0007	0.0002	0.0002	0.0001
0.75	-0.0032	-0.0060	-0.0007	-0.0059	-0.0059	-0.0062	0.0030	0.0002	0.0055	0.0003	0.0003
1.00	-0.0427	-0.0530	-0.0639	-0.0542	-0.0543	-0.0548	0.0121	0.0018	0.0091	0.0006	0.0005
1.25	0.0025	0.0068	0.0220	0.0066	0.0065	0.0065	0.0040	0.0003	0.0155	0.0001	0.0000
1.50	-0.0079	-0.0260	-0.0461	-0.0264	-0.0265	-0.0270	0.0191	0.0010	0.0191	0.0006	0.0005
1.75	-0.0025	0.0137	0.0339	0.0131	0.0131	0.0131	0.0156	0.0006	0.0208	0.0000	0.0000
2.00	-0.0247	-0.0300	-0.0459	-0.0294	-0.0295	-0.0300	0.0053	0.0000	0.0159	0.0006	0.0005
2.25	-0.0273	-0.0250	-0.0160	-0.0263	-0.0264	-0.0266	0.0007	0.0016	0.0106	0.0003	0.0002
2.50	-0.0124	-0.0140	-0.0125	-0.0134	-0.0135	-0.0138	0.0014	0.0002	0.0013	0.0004	0.0003

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Table 2 The displacement of the first story in two-story shear building obtained from various methods and the error values

Displacement u_1 (m)							Error				
t (sec)	Central Diff	Newmark Ave.	Wilson-ɗ	Proposed 4th	Proposed 5th	Exact	Central Diff	Newmark Ave.	Wilson-ɗ	Proposed 4th	Proposed 5th
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-	-	-	-	-
0.25	-0.1263	-0.2293	-0.2433	-0.2345	-0.2348	-0.2373	0.1110	0.0080	0.0060	0.0028	0.0025
0.50	0.0589	0.0156	0.0452	0.0237	0.0235	0.0182	0.0407	0.0026	0.0270	0.0055	0.0053
0.75	-0.1527	-0.1335	-0.1634	-0.1389	-0.1389	-0.1409	0.0118	0.0074	0.0225	0.0020	0.0020
1.00	0.0248	0.0565	0.0805	0.0563	0.0553	0.0548	0.0300	0.0017	0.0257	0.0015	0.0005
1.25	-0.1688	-0.1810	-0.2315	-0.1960	-0.1961	-0.1922	0.0234	0.0112	0.0393	0.0038	0.0039
1.50	-0.0682	-0.0966	-0.0670	-0.0959	-0.0963	-0.0986	0.0304	0.0020	0.0316	0.0027	0.0023
1.75	-0.0952	-0.1231	-0.1310	-0.1197	-0.1192	-0.1248	0.0296	0.0017	0.0062	0.0051	0.0056
2.00	0.0197	0.0223	0.0510	0.0354	0.0350	0.0256	0.0059	0.0033	0.0254	0.0098	0.0094
2.25	-0.0209	-0.0083	-0.0192	-0.0124	-0.0127	-0.0126	0.0083	0.0043	0.0066	0.0002	0.0001
2.50	-0.1020	-0.1055	-0.1379	-0.1196	-0.1207	-0.1146	0.0126	0.0091	0.0233	0.0050	0.0061

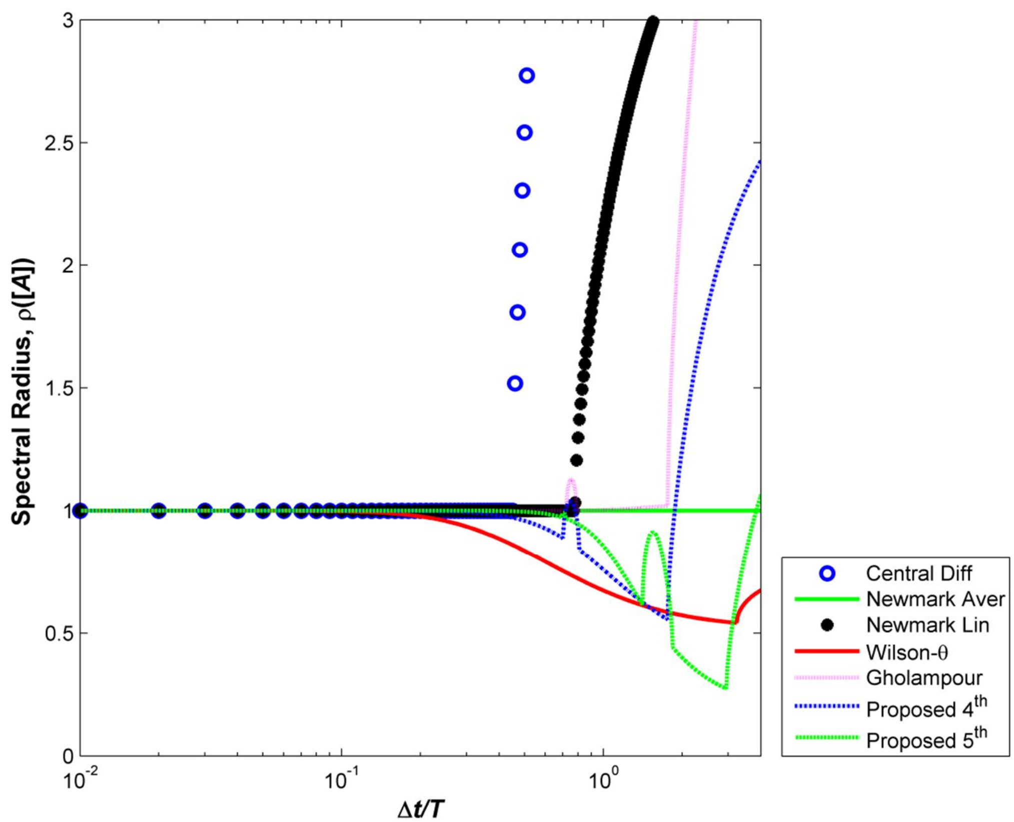

Fig.1 Spectral radii of integration schemes for μ_i=0.5.

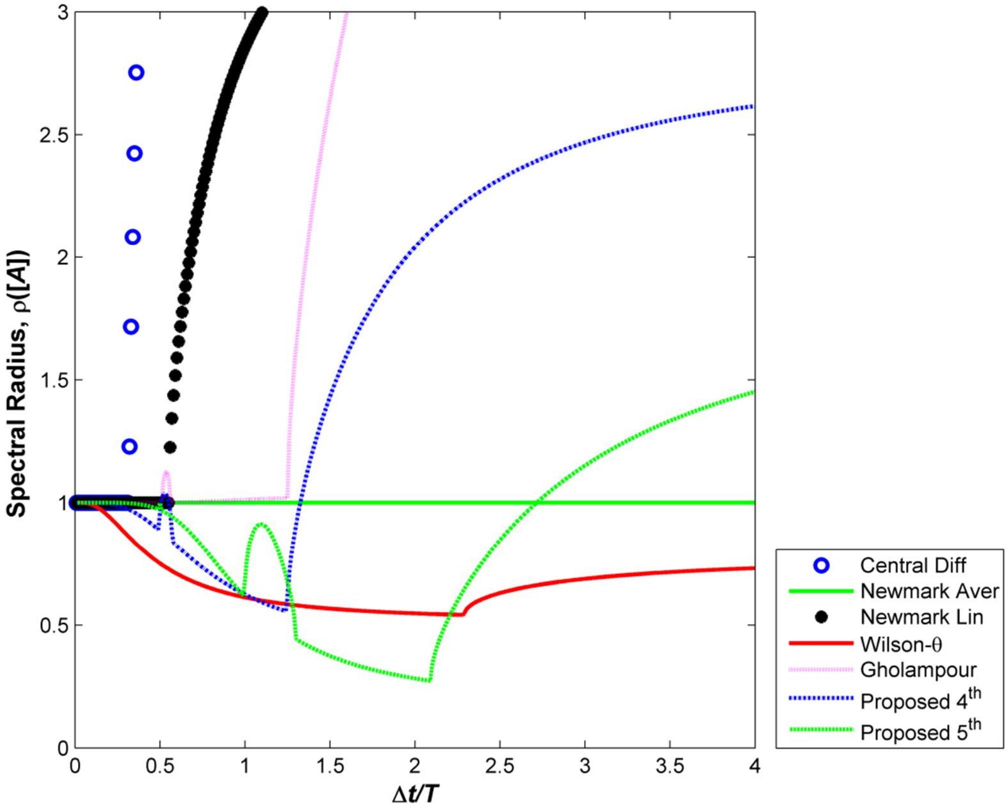

Fig.2 Spectral radii of integration schemes for μ_i=1.0.

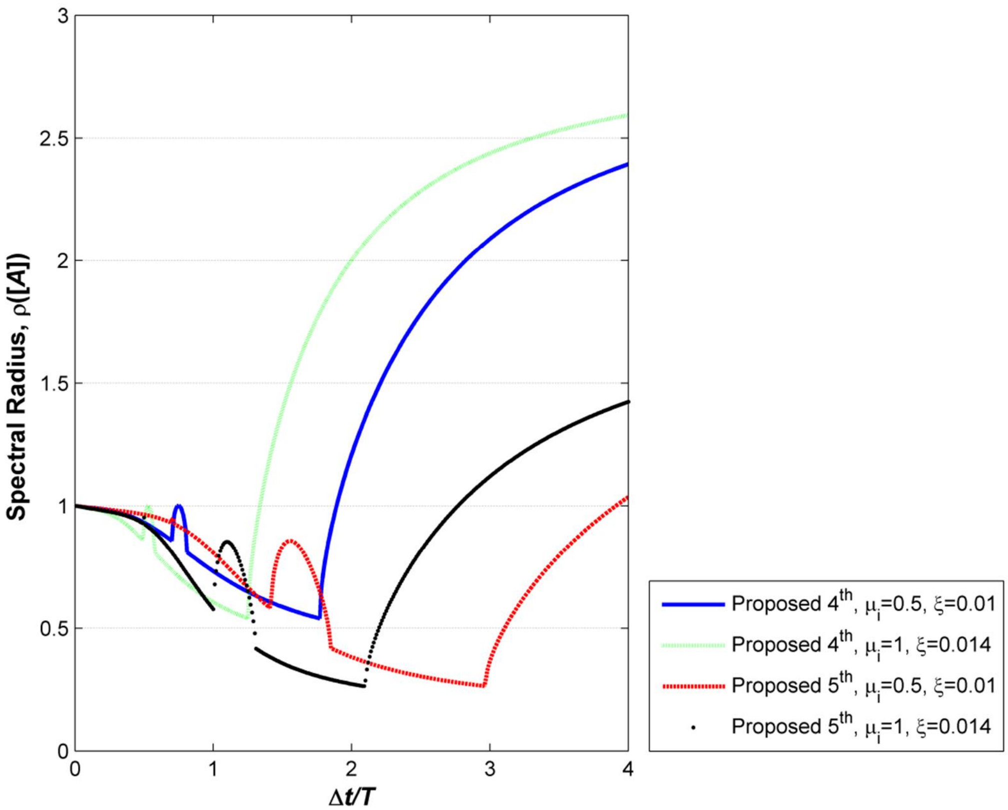

Fig.3 Spectral radii of the proposed schemes for different damping and stiffness ratios.

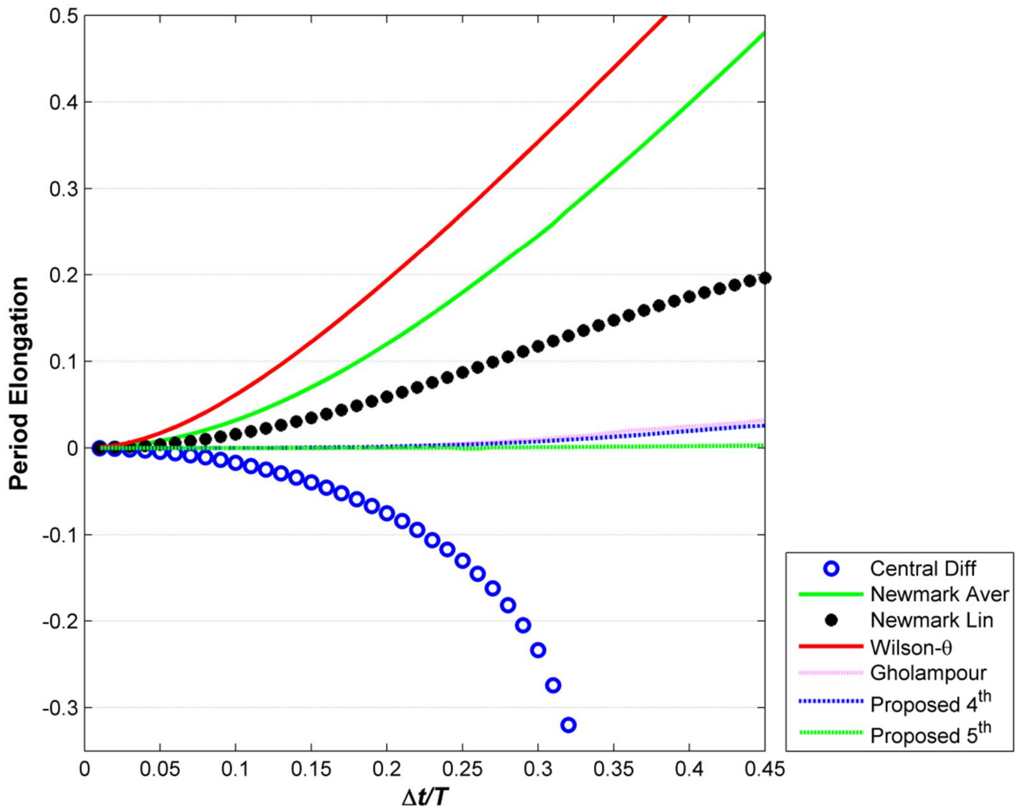

Fig. 4 Period elongation versus ϕt/*T* for the examined methods.

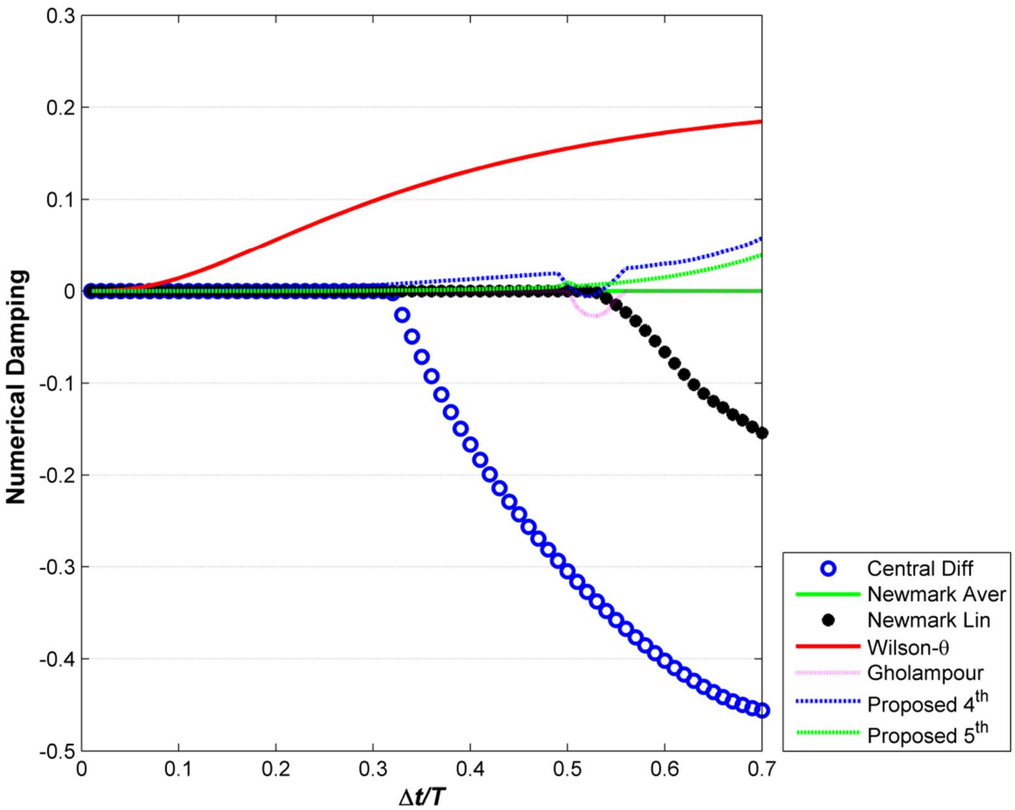

Fig. 5 Algorithmic damping versus φt/*T* for the examined methods.

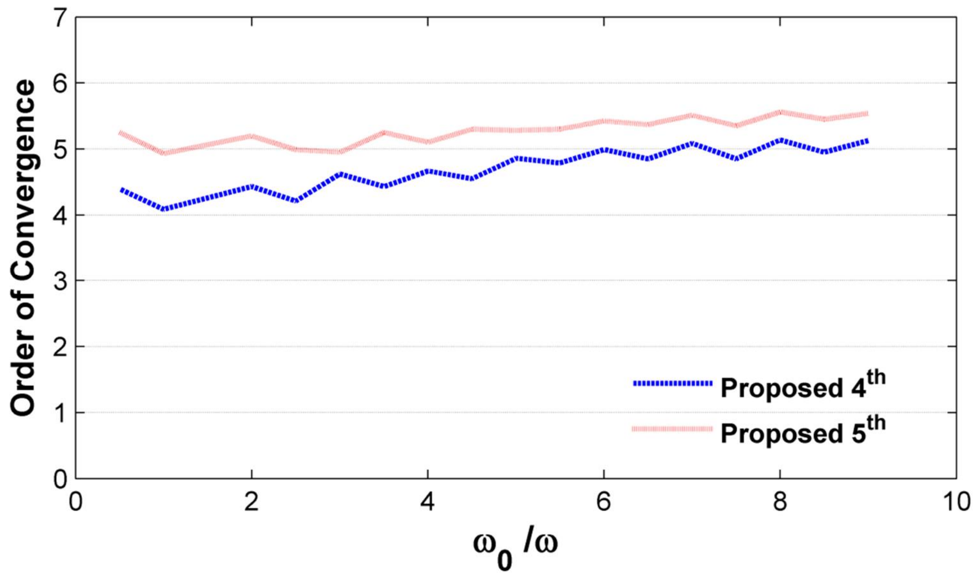

Fig. 6 Order of convergence for the proposed methods in the study.

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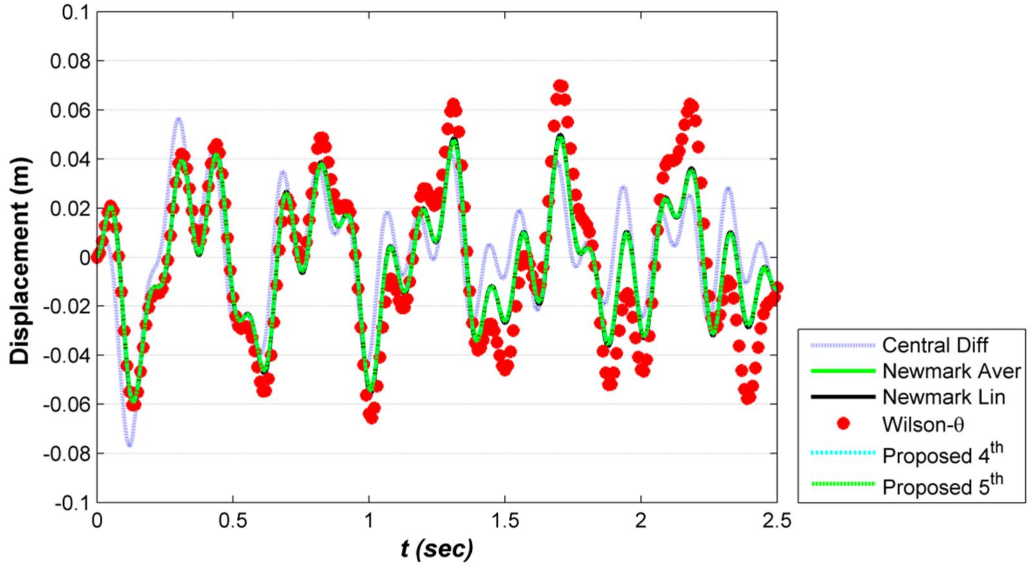

Fig. 7 Displacement responses of nonlinear single-degree-of-freedom system using different methods.

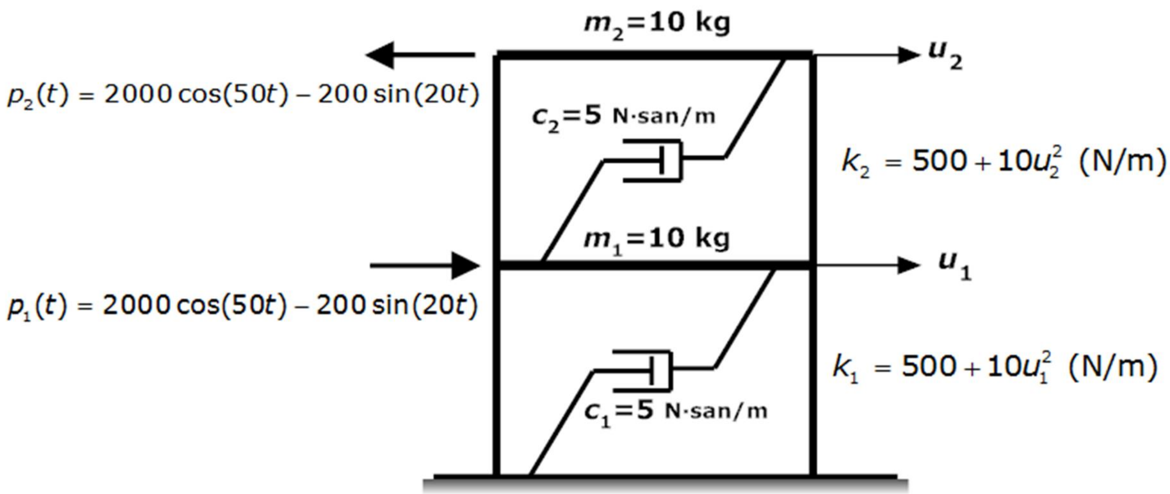

Fig. 8 Model of the shear building analyzed in Example 2.

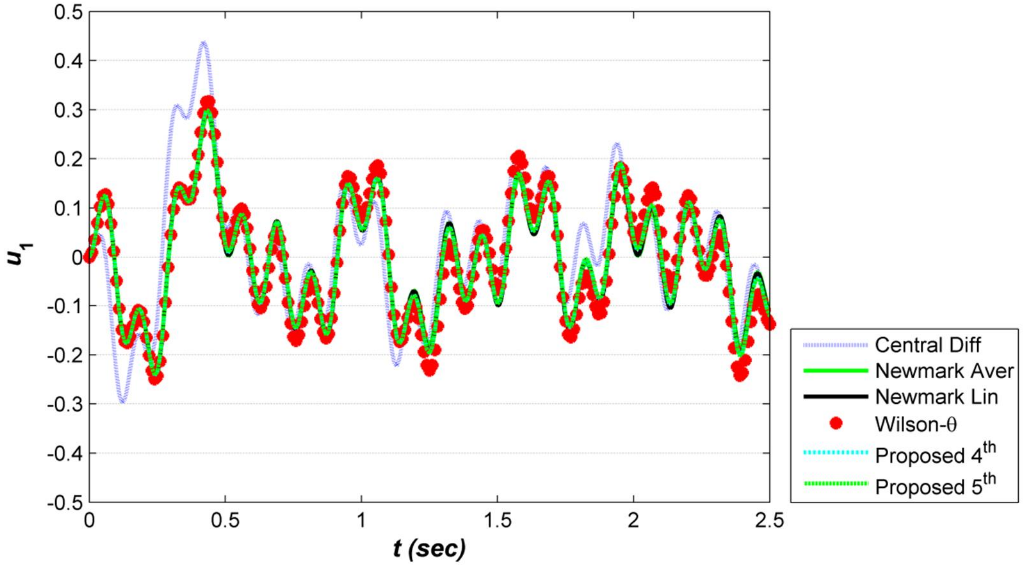

Fig. 9 First floor level displacement response of the nonlinear two-story shear frame.

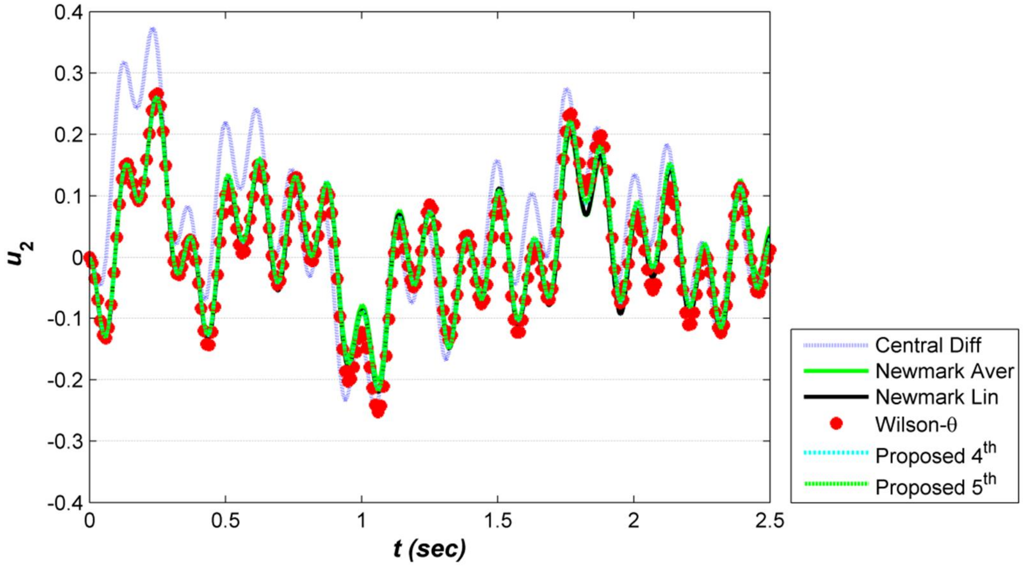

Fig. 10 Second floor level displacement response of the nonlinear two-story shear frame.

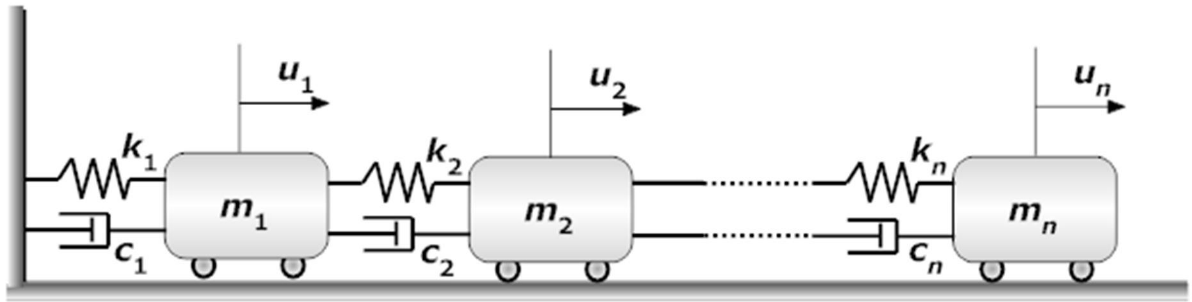

Fig. 11 Model of the n degree-of-freedom damped, spring-mass system analyzed in Example 3.

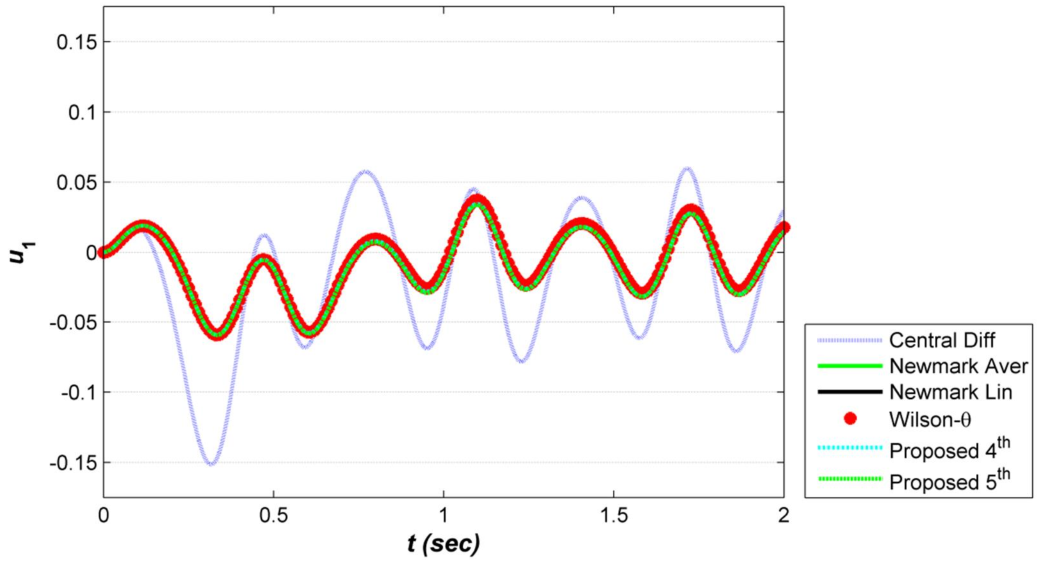

Fig. 12 The displacement solution of the first degree-of-freedom of the system with n=100.